Math 260 Exam #4 Key (Fall 2017)

1 Complete the square:

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 4x^2 + 2xy + 10y^2 = 4\left(x + \tfrac{1}{4}y\right)^2 + \tfrac{39}{4}y^2.$$

The expression at right can be seen to be never negative, and in order for it to equal zero it's necessary to have $x = -\frac{1}{4}y$ and $y = 0$; that is, $x = y = 0$ is necessary. Thus $\mathbf{x}^\top \mathbf{A}\mathbf{x} \leq \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$, and therefore $\mathbf{A}$ is positive definite.

2 With the Gram-Schmidt Orthogonalization Process we obtain the orthonormal basis

$$\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},\ \frac{1}{5\sqrt{3}}\begin{bmatrix} -1 \\ 7 \\ -5 \end{bmatrix} \right).$$

3 Let $\mathbf{w}_1 = 1$, $\mathbf{u}_2 = t$, $\mathbf{u}_3 = t^2$, $\mathbf{u}_4 = t^3$. Then by the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 1\,dt}(1) = t - \frac{1}{2},$$

$$\mathbf{w}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{u}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 = t^2 - \frac{\int_0^1 t^2\,dt}{\int_0^1 1\,dt}(1) - \frac{\int_0^1 t^2\left(t - \frac{1}{2}\right)dt}{\int_0^1 \left(t - \frac{1}{2}\right)^2 dt}\left(t - \tfrac{1}{2}\right) = t^2 - t + \tfrac{1}{6},$$

and

$$\begin{aligned}
\mathbf{w}_4 &= \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{u}_4, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \frac{\langle \mathbf{u}_4, \mathbf{w}_3 \rangle}{\langle \mathbf{w}_3, \mathbf{w}_3 \rangle}\mathbf{w}_3 \\
&= t^3 - \frac{1/4}{1}(1) - \frac{3/40}{1/12}(t - 1/2) - \frac{1/120}{1/180}(t^2 - t - 1/6) \\
&= t^3 - \frac{3}{2}t^2 + \frac{3}{5}t + \frac{1}{5}.
\end{aligned}$$

So

$$\left\{1,\ t - \frac{1}{2},\ t^2 - t + \frac{1}{6},\ t^3 - \frac{3}{2}t^2 + \frac{3}{5}t + \frac{1}{5}\right\}$$

is an orthogonal basis.

4 Expand down first column, say:

$$4\begin{vmatrix} a & -13 \\ 0 & a \end{vmatrix} + \begin{vmatrix} -2 & 3b \\ a & -13 \end{vmatrix} = 4a^2 + (26 - 3ab) = 4a^2 - 3ab + 26.$$

5 Perform the row operations $r_2 + r_3 \to r_3$, $2r_2 + r_4 \to r_4$, and $2r_2 + r_1 \to r_1$ to obtain

$$\begin{aligned}
\begin{vmatrix} 0 & 4 & 15 & 2 \\ 1 & 1 & 6 & 3 \\ 0 & 1 & 7 & 2 \\ 0 & t+2 & 18 & 11 \end{vmatrix} &= -\begin{vmatrix} 4 & 15 & 2 \\ 1 & 7 & 2 \\ t+2 & 18 & 11 \end{vmatrix} \\
&= -\left(4\begin{vmatrix} 7 & 2 \\ 18 & 11 \end{vmatrix} - \begin{vmatrix} 15 & 2 \\ 18 & 11 \end{vmatrix} + (t+2)\begin{vmatrix} 15 & 2 \\ 7 & 2 \end{vmatrix}\right) \\
&= -16t - 67.
\end{aligned}$$

6 We have

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -4 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = -20.$$

Now, with $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ and $\mathbf{b} = [0\ 1\ 0\ 4]^\top$, we have

$$x = \frac{\det(\mathbf{b}\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4)}{\det(\mathbf{A})} = -\frac{1}{20}\begin{vmatrix} 0 & 2 & -3 & 5 \\ 1 & 1 & -4 & -1 \\ 0 & 1 & 1 & 1 \\ 4 & -1 & -1 & 1 \end{vmatrix} = -\frac{1}{20}(-110) = \frac{11}{2}.$$

$$y = \frac{\det(\mathbf{a}_1\ \mathbf{b}\ \mathbf{a}_3\ \mathbf{a}_4)}{\det(\mathbf{A})} = -\frac{1}{20}\begin{vmatrix} 1 & 0 & -3 & 5 \\ 2 & 1 & -4 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 4 & -1 & 1 \end{vmatrix} = -\frac{152}{20} = -\frac{38}{5},$$

$$z = \frac{\det(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}\ \mathbf{a}_4)}{\det(\mathbf{A})} = -\frac{1}{20}\begin{vmatrix} 1 & 2 & 0 & 5 \\ 2 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 4 & 1 \end{vmatrix} = \frac{2}{20} = \frac{1}{10},$$

$$w = \frac{\det(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b})}{\det(\mathbf{A})} = -\frac{1}{20}\begin{vmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -4 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 4 \end{vmatrix} = \frac{40}{20} = 2.$$

Thus by Cramer's Rule the solution to the system is

$$\begin{bmatrix} 11/2 \\ -38/5 \\ 1/10 \\ 2 \end{bmatrix}.$$

7 All the 3×3 determinants formed from $\mathbf{H}$ equal 0, and so the rank of $\mathbf{H}$ must be less than 3. On the other hand

$$\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} \neq 0,$$

so the matrix

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 3 & 9 \end{bmatrix}$$

has rank 2. This means the column vectors of this matrix are linearly independent, and since they are the last two columns of $\mathbf{H}$, it follows that the rank of $\mathbf{H}$ is at least 2. Therefore $\operatorname{rank}(\mathbf{H}) = 2$.