Math 260 Exam #3 Key (Fall 2017)

1 Find a and b such that
$$a(1,1)+b(-1,3)=(0,1),$$
so $a-b=0$ and $a+3b=1$. We find $a=b=1/4$. Now, by linearity,
$$L(0,1)=L\big(\tfrac{1}{4}(1,1)+\tfrac{1}{4}(-1,3)\big)=\tfrac{1}{4}L(1,1)+\tfrac{1}{4}L(-1,3)=\tfrac{1}{4}(2,-1)+\tfrac{1}{4}(1,2)=\big(\tfrac{3}{4},\tfrac{1}{4}\big).$$

2 Suppose $\sum_{k=1}^{n} c_k\mathbf{v}_k=\mathbf{0}$. Then
$$\mathbf{0}=L(\mathbf{0})=L\left(\sum_{k=1}^{n} c_k\mathbf{v}_k\right)=\sum_{k=1}^{n} c_kL(\mathbf{v}_k)=\sum_{k=1}^{n} c_k\mathbf{w}_k$$
by linearity properties, and since $\mathbf{w}_1,\ldots,\mathbf{w}_n$ are linearly independent we conclude that $c_k=0$ for all k.

3 By the Rank-Nullity Theorem, $\dim(\operatorname{Img} L)+\dim(\operatorname{Ker} L)=\dim V$. Since $\operatorname{Img} L$ is a subspace of W, $\dim(\operatorname{Img} L)\leq\dim W$. Thus $\dim(\operatorname{Img} L)<\dim V$, and it follows that $\dim(\operatorname{Ker} L)>0$. That is, $\dim(\operatorname{Ker} L)\geq 1$, and we conclude that $\operatorname{Ker} L\neq\{\mathbf{0}\}$.

4 Let
$$\mathbf{v}_1^\top=[1,1,-2,3,4],\quad \mathbf{v}_2^\top=[1,0,0,2,0,],\quad \mathbf{v}_3^\top=[0,1,0,1,0],$$
and let $\mathbf{A}$ be the 3×5 matrix with row vectors $\mathbf{v}_1^\top$, $\mathbf{v}_2^\top$, $\mathbf{v}_3^\top$. Three column vectors of $\mathbf{A}$ are
$$\begin{bmatrix}3\\2\\1\end{bmatrix},\quad \begin{bmatrix}1\\1\\0\end{bmatrix},\quad \begin{bmatrix}4\\0\\0\end{bmatrix},$$
which are linearly independent in $\mathbb{R}^3$, and so $\operatorname{rank}\mathbf{A}=3$. Let U be the subspace in question, so
$$U=\{\mathbf{x}\in\mathbb{R}^5:\mathbf{v}_1\cdot\mathbf{x}=0,\ \mathbf{v}_2\cdot\mathbf{x}=0,\ \mathbf{v}_3\cdot\mathbf{x}=0\}=\{\mathbf{x}\in\mathbb{R}^5:\mathbf{A}\mathbf{x}=\mathbf{0}\}.$$
By the Matrix Rank-Nullity Theorem
$$\dim U=\dim(\operatorname{Nul}\mathbf{A})=\operatorname{nullity}\mathbf{A}=\dim\mathbb{R}^5-\operatorname{rank}\mathbf{A}=5-3=2.$$

5 If $\mathcal{E}_4$ and $\mathcal{E}_3$ are the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, then
$$[L]_{\mathcal{E}_4\mathcal{E}_3}=\Big[\big[L(\mathbf{e}_1)\big]_{\mathcal{E}_3}\ \cdots\ \big[L(\mathbf{e}_4)\big]_{\mathcal{E}_3}\Big]=\begin{bmatrix}0&0&2&0\\0&0&0&0\\-2&0&0&0\end{bmatrix}.$$

6 Let $\mathbf{v}\in V$. Now, $(P\circ P)(\mathbf{v})=P(\mathbf{v})$ shows that $P(\mathbf{v})\in\operatorname{Img} P$. Also
$$P(\mathbf{v}-P(\mathbf{v}))=P(\mathbf{v})-P(P(\mathbf{v}))=P(\mathbf{v})-P(\mathbf{v})=\mathbf{0},$$
which shows $\mathbf{v}-P(\mathbf{v})\in\operatorname{Ker} P$. Thus
$$\mathbf{v}=[\mathbf{v}-P(\mathbf{v})]+P(\mathbf{v})\in\operatorname{Ker} P+\operatorname{Img} P,$$
so that $V\subseteq\operatorname{Ker} P+\operatorname{Img} P$. That $\operatorname{Ker} P+\operatorname{Img} P\subseteq V$ follows from the usual closure properties of a vector space, and therefore $V=\operatorname{Ker} P+\operatorname{Img} P$.

7 Suppose $L(x, y) = (0, 0)$. Then $2x + y = 0$ and $3x - 5y = 0$, which are only satisfied if $x = y = 0$. Thus $\operatorname{Ker} L = \{(0, 0)\}$, which implies L is injective, hence surjective, hence bijective, and therefore invertible.

8 Setting $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2)$, the transition matrix is $\mathbf{I}_{\mathcal{BC}} = \big[[\mathbf{v}_1]_{\mathcal{C}} \;\; [\mathbf{v}_2]_{\mathcal{C}}\big]$. Here $[\mathbf{v}_1]_{\mathcal{C}} = [a \;\; b]^\top$ for a, b such that $a\mathbf{w}_1 + b\mathbf{w}_2 = \mathbf{v}_1$, and $[\mathbf{v}_2]_{\mathcal{C}} = [c \;\; d]^\top$ for c, d such that $c\mathbf{w}_1 + d\mathbf{w}_2 = \mathbf{v}_2$. Solving the two systems of equations

$$\begin{cases} a + b = 1 \\ -a + b = 1 \end{cases} \qquad \begin{cases} c + d = 2 \\ -c + d = 0 \end{cases}$$

yields $a = 0$, $b = 1$, $c = 1$, and $d = 1$, and so

$$\mathbf{I}_{\mathcal{BC}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$