# Math 260 Exam #2 Key (Fall 2017)

**1** If $x_k - 2y_k + 3z_k = 0$ for $k = 1, 2$, then

$$(x_1 + x_2) - 2(y_1 + y_2) + 3(z_1 + z_2) = (x_1 - 2y_1 + 3z_1) + (x_2 - 2y_2 + 3z_2) = 0.$$

This shows that if $\mathbf{v}_1 = [x_1, y_1, z_1]$ and $\mathbf{v}_2 = [x_2, y_2, z_2]$ belong to the set, then so does

$$\mathbf{v}_1 + \mathbf{v}_2 = [x_1 + x_2, y_1 + y_2, z_1 + z_2].$$

Also if $x - 2y + 3z = 0$, then

$$(cx) - 2(cy) + 3(cz) = 0,$$

which shows that if $\mathbf{v} = [x, y, z]$ belongs so the set, then so does $c\mathbf{v} = [cx, cy, cz]$. The set is therefore closed under vector addition and scalar multiplication, and since $[0, 0, 0]$ clearly belongs to the set, the set is indeed a subspace of $\mathbb{R}^3$.

**2** Note that $[2, 1]$ and $[2, -1]$ belong to the set, but $[2, 1] + [2, -1] = [4, 0]$ does not since $4 - 2(0)^2 = 4 \neq 0$. The set is not closed under vector addition and therefore is not a subspace of $\mathbb{R}^2$.

**3** By definition

$$W = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}.$$

Clearly $\mathbf{0} \in W$. Suppose $\mathbf{w}_1, \mathbf{w}_2 \in W$, and let $\mathbf{v} \in V$ be arbitrary. Then

$$(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{v} = \mathbf{w}_1 \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v} = 0 + 0 = 0,$$

and so $\mathbf{w}_1 + \mathbf{w}_2 \in W$. So $W$ is closed under vector addition. Next, for any $\mathbf{w} \in W$ and $c \in \mathbb{R}$ we find, for any $\mathbf{v} \in V$, that $(c\mathbf{w}) \cdot \mathbf{v} = c(\mathbf{w} \cdot \mathbf{v}) = (c)(0) = 0$, and hence $c\mathbf{w} \in W$. So $W$ is closed under scalar multiplication. Therefore $W$ is a subspace of $\mathbb{R}^n$.

**4** Suppose

$$x_1[1, 2, 0] + x_2[1, 3, -1] + x_3[-1, 1, 1] = [0, 0, 0].$$

Then we obtain the system

$$\begin{cases} x_1 + \phantom{3}x_2 - x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ \phantom{2x_1} -x_2 + x_3 = 0 \end{cases}$$

The last equation gives $x_3 = x_2$, which can be used to go on to find that $x_1 = x_2 = x_3 = 0$. Therefore the vectors are linearly independent.

**5** We must show that $P = \{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : t_1, t_2 \in [0, 1]\}$ is a convex set. Suppose $\mathbf{p}, \mathbf{q} \in P$, so

$$\mathbf{p} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{q} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

for $s_1, s_2, t_1, t_2 \in [0, 1]$. Fix $\mathbf{x} \in [\mathbf{p}, \mathbf{q}]$, so for some $u \in [0, 1]$ we have

$$\mathbf{x} = (1 - u)\mathbf{p} + u\mathbf{q} = (1 - u)(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + u(t_1\mathbf{v}_1 + t_2\mathbf{v}_2).$$

Rearranging gives

$$\mathbf{x} = [(1 - u)s_1 + ut_1]\mathbf{v}_1 + [(1 - u)s_2 + ut_2]\mathbf{v}_2,$$

and since $0 \leq s_1, s_2, t_1, t_2 \leq 1$ it follows that

$$0 \leq (1 - u)s_1 + ut_1 \leq (1 - u)(1) + (u)(1) = (1 - u) + u = 1$$

and similarly $0 \le (1-u)s_2 + ut_2 \le 1$. Hence $\mathbf{x} \in P$, which shows that $[\mathbf{p}, \mathbf{q}] \subseteq P$, and therefore $P$ is convex.

**6** Find $x_1$ and $x_2$ such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{x}$, This results in the system

$$\begin{cases} 2x_1 - x_2 = \phantom{-}4 \\ x_1 \phantom{- x_2} = -3 \end{cases}$$

Solving yields $x_1 = -3$ and $x_2 = -10$. If we define $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, then the coordinates of $\mathbf{x}$ with respect to $\mathbf{u}_1, \mathbf{u}_2$ are otherwise known as the $\mathcal{B}$-coordinates of $\mathbf{x}$, denoted by $[\mathbf{x}]_{\mathcal{B}} = [-3, -10]$.

**7** We have $x = 2y - 3z$, so (letting $s = y$ and $t = z$) we obtain

$$V = \left\{ \begin{bmatrix} 2y-3z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The vectors in this spanning set for $V$ can be shown to be linearly independent, and therefore a basis for $V$ is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

**8** Let $f(x) = e^t$ and $g(t) = \ln t$. Suppose $c_1 f + c_2 g = 0$ on $(0, \infty)$. Then in particular we have $c_1 f(1) + c_2 g(1) = 0$ and $c_1 f(2) + c_2 g(2) = 0$. The first equation gives $c_1 e + c_2 \ln(1) = 0$, and hence $c_1 = 0$. The second equation then becomes $c_2 g(2) = 0$, or $c_2 \ln 2 = 0$, and hence $c_2 = 0$. Therefore $f$ and $g$ are linearly independent on $(0, \infty)$.

**9** With the elementary row operations $-3r_1 + r_2$, $r_1 + r_3$, and $-3r_1 + r_4$ we obtain

$$\mathbf{M} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of $\mathbf{M}$ is equal to the number of pivots in a row-equivalent row-echelon form, and so $\operatorname{rank}(\mathbf{M}) = 2$.

**10** In terms of column vectors we have $\mathbf{A} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ and $\mathbf{B} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_r]$. Now, since

$$\mathbf{AB} = [\mathbf{Ab}_1 \ \cdots \ \mathbf{Ab}_r],$$

the $\ell$th column vector of $\mathbf{AB}$ is $\mathbf{Ab}_\ell$. Set $\mathbf{b}_\ell = [b_{1\ell} \ \cdots \ b_{n\ell}]^\top$. Then, letting $a_{ij}$ denote the $ij$-entry of $\mathbf{A}$ in general,

$$\mathbf{Ab}_\ell = \begin{bmatrix} \sum_{j=1}^n a_{1j} b_{j\ell} \\ \vdots \\ \sum_{j=1}^n a_{mj} b_{j\ell} \end{bmatrix} = \sum_{j=1}^n \begin{bmatrix} a_{1j} b_{j\ell} \\ \vdots \\ a_{mj} b_{j\ell} \end{bmatrix} = \sum_{j=1}^n b_{j\ell} \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = \sum_{j=1} b_{j\ell} \mathbf{a}_j.$$

Thus each of the column vectors of $\mathbf{AB}$ is a linear combination of the column vectors of $\mathbf{A}$, so that the column space of $\mathbf{AB}$ is a subset of the column space of $\mathbf{A}$: $\operatorname{Col}(\mathbf{AB}) \subseteq \operatorname{Col}(\mathbf{A})$. Therefore

$$\operatorname{rank}(\mathbf{AB}) = \dim[\operatorname{Col}(\mathbf{AB})] \leq \dim[\operatorname{Col}(\mathbf{A})] = \operatorname{rank}(\mathbf{A}).$$

**11** The image of the line $x = c$ under $F$ is

$$\{F(c, y) : y \in \mathbb{R}\} = \{e^{-c}[\sin y, \cos y] : y \in \mathbb{R}\}.$$

Letting $u = e^{-c} \sin y$ and $v = e^{-c} \cos y$, we have $u^2 + v^2 = e^{-c}$. Thus the image of $x = c$ under $F$ is a circle with center $(0, 0)$ and radius $e^{-c}$.

Next, the image of $y = d$ under $F$ is

$$\{F(x, d) : x \in \mathbb{R}\} = \{e^{-x}[\sin d, \cos d] : x \in \mathbb{R}\}.$$

The range of $e^{-x}$ for $x \in \mathbb{R}$ is $(0, \infty)$, and so the image of $y = d$ under $F$ is equivalently written as

$$\{x[\sin d, \cos d] : x > 0\},$$

and since $[\sin d, \cos d] \neq [0, 0]$, we find the image to be an open ray emanating from the origin.