Math 260 Exam #1 Key (Fall 2017)

1a Both $\pm\mathbf{u}/\|\mathbf{u}\|$ will work, where $\mathbf{u}/\|\mathbf{u}\| = \frac{1}{\sqrt{26}}[3,-1,4]$.

1b First find any $\mathbf{v} = [v_1, v_2, v_3]$ such that $\mathbf{u}\cdot\mathbf{v} = 3v_1 - v_2 + 4v_3 = 0$. Clearly letting $v_1 = 1$, $v_2 = 3$, and $v_3 = 0$ will work, so $\mathbf{v} = [1,3,0]$. Now either $\pm 4\mathbf{v}/\|\mathbf{v}\|$ will work, where

$$\frac{4}{\|\mathbf{v}\|}\mathbf{v} = \frac{4}{\sqrt{10}}[1,3,0].$$

Other answers are possible.

2a We have

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{10}{6}[-2,-1,1] = \left[-\tfrac{10}{3}, -\tfrac{5}{3}, \tfrac{5}{3}\right].$$

2b By definition

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{10}{\sqrt{20}\sqrt{6}} = \frac{5}{\sqrt{30}} = \frac{\sqrt{30}}{6}.$$

3 The constant is 4:

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= (\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v})\\ &= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.\end{aligned}$$

4 $\mathbf{x}(t) = \mathbf{p} + t(\mathbf{q}-\mathbf{p}) = [1,0,-1] + t[1,2,-2] = [1+t, 2t, -1-2t]$ for $t \in \mathbb{R}$.

5 The plane consists of all points $\mathbf{x} = [x,y,z]$ such that $\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) = 0$, so that

$$[-3,2,8]\cdot[x-1/2, y, z-2] = 0,$$

and finally

$$-3x + 2y + 8z = \tfrac{29}{2}.$$

6 Two points determine the line, and they can be found as solutions to the system

$$\begin{cases} x - y + z = 2\\ 2x - 3y + z = 6\end{cases}$$

The first equation gives $z = 2-x+y$, which when put into the second equation gives $x = 2y+4$. Putting this back into $z = 2-x+y$ gives $z = -y-2$. The solution set of the system is

$$S = \big\{[2y+4, y, -y-2] : y \in \mathbb{R}\big\}.$$

Thus a parametric equation for the line of intersection of the two planes is

$$\mathbf{x}(t) = [2t + 4, t, -t - 2].$$

7 We have

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = \mathbf{A}\begin{bmatrix} 7 \\ -1 \\ a-2 \end{bmatrix} = \begin{bmatrix} 12 \\ 2a + 22 \end{bmatrix}.$$

8 We have $\mathbf{A}^3 - \mathbf{A} = -\mathbf{I}$, so that $\mathbf{A}(\mathbf{A}^2 - \mathbf{I}) = -\mathbf{I}$, and hence $\mathbf{A}(-\mathbf{A}^2 + \mathbf{I}) = \mathbf{I}$. Similarly $(-\mathbf{A}^2 + \mathbf{I})\mathbf{A} = \mathbf{I}$. This shows that $-\mathbf{A}^2 + \mathbf{I}$ is an inverse for $\mathbf{A}$, and therefore $\mathbf{A}$ is invertible.

9a Since $\mathbf{A}$ is similar to $\mathbf{B}$ there exists invertible $\mathbf{T}$ such that $\mathbf{B} = \mathbf{TAT}^{-1}$. Now, $\mathbf{T}^{-1}$ is an invertible matrix such that

$$\mathbf{T}^{-1}\mathbf{BT} = \mathbf{T}^{-1}(\mathbf{TAT}^{-1})\mathbf{T} = (\mathbf{T}^{-1}\mathbf{T})\mathbf{A}(\mathbf{T}^{-1}\mathbf{T}) = \mathbf{IAI} = \mathbf{A},$$

and therefore $\mathbf{B}$ is similar to $\mathbf{A}$.

9b Suppose $\mathbf{A}$ is invertible, which is to say $\mathbf{A}^{-1}$ exists. Now, since $\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$ and $\mathbf{AA}^{-1} = \mathbf{I}$,

$$\mathbf{B}(\mathbf{TA}^{-1}\mathbf{T}^{-1}) = (\mathbf{TAT}^{-1})(\mathbf{TA}^{-1}\mathbf{T}^{-1}) = \mathbf{TA}(\mathbf{T}^{-1}\mathbf{T})\mathbf{A}^{-1}\mathbf{T}^{-1} = \mathbf{TAA}^{-1}\mathbf{T}^{-1} = \mathbf{TT}^{-1} = \mathbf{I},$$

and similarly $(\mathbf{TA}^{-1}\mathbf{T}^{-1})\mathbf{B} = \mathbf{I}$. This shows that $\mathbf{TA}^{-1}\mathbf{T}^{-1}$ is the inverse for $\mathbf{B}$, and therefore $\mathbf{B}$ is invertible.

If we next suppose that $\mathbf{B}$ is invertible, then since $\mathbf{B}$ is similar to $\mathbf{A}$ by part (a), a symmetrical argument (i.e. one in which we interchange the roles of $\mathbf{A}$ and $\mathbf{B}$ in the previous paragraph) shows that $\mathbf{A}$ must be invertible.

10 Performing row operations on

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

until $\mathbf{I}_3$ is obtained on the left side (the chosen series of steps can vary), we find that

$$\mathbf{C}^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

11 The corresponding augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -11 & -27 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & 0 & -2/7 \end{array}\right].$$

The third equation now states that $0 = -2/7$, which is a contradiction. Therefore the system has no solution.

12 From the 2nd equation: $x = 2z - y$. Putting this into the 1st equation yields $y = 3z - 1$. Thus $x = 2z - y = 1 - z$, and solution set is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 1 - z \text{ and } y = 3z - 1 \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} 1-t \\ 3t-1 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$