1.6.1a It can help to get the parametric equation for L_2. Letting $z = 1$ gives $x = 5$ and

$$y = \frac{z-1}{2} + 4 = \frac{1-1}{2} + 4 = 4,$$

so $p_1 = (5, 4, 1)$ is one point on L_2. Letting $z = 3$ gives $x = 5$ and

$$y = \frac{z-1}{2} + 4 = \frac{3-1}{2} + 4 = 5,$$

so $p_2 = (5, 5, 3)$ is another point on L_2. Letting

$$\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1 = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

then L_2 is given by

$$\mathbf{q}(t) = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R},$$

while L_1 is given by

$$\mathbf{p}(s) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

We must find $s, t \in \mathbb{R}$ such that $\mathbf{p}(s) = \mathbf{q}(t)$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

This gives rise to the system of equations

$$\begin{cases} 2s + 1 = 5 \\ s + 1 = t + 4 \\ -s + 1 = 2t + 1 \end{cases}$$

which has the unique solution $(s, t) = (2, -1)$. That is,

$$\mathbf{q}(-1) = \mathbf{p}(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix},$$

so $(5, 3, -1)$ is the point of intersection.

1.6.1b The plane certainly contains all points that L_1 and L_2 contain, such as

$$\mathbf{a} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}.$$

Letting

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \mathbf{c} - \mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

a parametric equation for the plane is $\mathbf{p}(s,t) = \mathbf{a} + s\mathbf{u} + t\mathbf{v}$, $(s,t) \in \mathbb{R}^2$. That is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} + s\begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \tag{1}$$

To find the algebraic equation, we obtain from (1) the system

$$\begin{cases} x = 5 - 4s \\ y = 3 - 2s + t \\ z = -1 + 2s + 2t \end{cases}$$

From the system's first and second equations we have

$$s = \frac{5-x}{4} \quad \text{and} \quad t = y - 3 + 2s = y - 3 + \frac{5-x}{4},$$

Substituting these into the third equation then yields

$$z = -1 + 2s + 2t = -1 + 2\left(\frac{5-x}{4}\right) + 2\left(y - 3 + \frac{5-x}{4}\right) = \frac{1}{2} + 2y - \frac{3}{2}x,$$

or

$$3x - 4y + 2z = 1,$$

which is the algebraic equation of the plane.

1.6.2a Geometrically, the plane P may be characterized as the set of all points $p = (x,y,z) \in \mathbb{R}^3$ such that the vector $\overrightarrow{ap}$ is orthogonal to $\mathbf{n}$, which is to say $\overrightarrow{ap} \cdot \mathbf{n} = 0$. Now,

$$0 = \overrightarrow{ap} \cdot \mathbf{n} = \begin{bmatrix} x-5 \\ y-1 \\ z-3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = (x-5) - 4(y-1) + 2(z-3),$$

and thus $x - 4y + 2z = 7$ is the algebraic equation of P.

1.6.2b From the algebraic equation we have $x = 7 - 4y + 2z$, and so P may be characterized as the set of vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 - 4y + 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix};$$

that is, P is given as the parametric equation

$$\mathbf{p}(s,t) = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}.$$

2.2.1a We have

$$\mathbf{x}^\top\mathbf{x} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = [14],$$

which in practice is identified with the scalar 14.

2.2.1b We have

$$\mathbf{x}\mathbf{x}^\top = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 6 \\ -3 & 1 & -2 \\ 6 & -2 & 4 \end{bmatrix}$$

2.2.1c We have

$$\mathbf{AC} = \begin{bmatrix} -2 & -9 \\ -12 & 3 \\ 9 & 7 \end{bmatrix}$$

2.5.1 The corresponding augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -4 & 1 & -20 \\ 0 & -1 & 5 & -5 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -1 & 5 & -5 \\ 0 & -4 & 1 & -20 \end{array}\right]$$

$$\xrightarrow{-4r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -1 & 5 & -5 \\ 0 & 0 & -19 & 0 \end{array}\right].$$

We have obtained the equivalent system of equations

$$\begin{cases} x + 2y - z = 9 \\ -y + 5z = -5 \\ -19z = 0 \end{cases}$$

From the third equation we have $z = 0$, which when put into the second equation yields $-y = -5$, or $y = 5$. Finally, from the first equation we obtain.

$$x + 2(5) - 0 = 9 \;\;\Rightarrow\;\; x = -1.$$

Therefore the sole solution to the system is $[-1, 5, 0]^\top$.

2.5.2 The corresponding augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & -2 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & -2 \end{array}\right] \xrightarrow[6r_1+r_3\to r_3]{2r_1+r_2\to r_2} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 3 & -3 & 2 \\ 0 & 6 & -6 & 4 \end{array}\right] \xrightarrow{-2r_2+r_3\to r_3} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have obtained the equivalent system of equations

$$\begin{cases} x - z = 1 \\ 3y - 3z = 2, \end{cases}$$

giving $x = z + 1$ and $y = z + \frac{2}{3}$. Any ordered triple (x, y, z) that satisfies the original system must be of the form

$$(z + 1, 3z + \tfrac{2}{3}, z)$$

for some $z \in \mathbb{R}$, and therefore the solution set is

$$\left\{(z + 1, 3z + \tfrac{2}{3}, z) : z \in \mathbb{R}\right\}.$$

In terms of column vectors, we have

$$\begin{bmatrix} z+1 \\ z+\frac{2}{3} \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and so solution set is

$$\left\{ \frac{1}{3}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ :\ t \in \mathbb{R} \right\},$$

which is a line in $\mathbb{R}^3$.

2.5.3 $\left\{\left[\frac{7}{4} - \frac{7}{4}w, -\frac{7}{4} + \frac{3}{4}w, \frac{9}{4} + \frac{3}{4}w, w\right]^\top : w \in \mathbb{R}\right\}.$

2.5.4 We employ the same sequence of elementary row operations on both $\mathbf{A}$ and $\mathbf{I}_3$, as follows.

$$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 7 \\ -1 & 2 & 2 & 3 & 1 \\ 4 & -8 & -3 & -2 & 6 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_1} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 1 \\ 3 & -6 & -1 & 1 & 7 \\ 4 & -8 & -3 & -2 & 6 \end{array}\right] \xrightarrow[4r_1+r_3\to r_3]{3r_1+r_2\to r_2}$$

$$\left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 5 & 10 & 10 \\ 0 & 0 & 5 & 10 & 10 \end{array}\right] \xrightarrow[r_1/5\to r_1]{-r_2+r_3\to r_3} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We now have the equivalent system of equations

$$\begin{cases} -x + 2y + 2z + 3w = 1 \\ z + 2w = 2 \end{cases}$$

giving $z = 2 - 2w$ and $x = 2y - w + 3$ for $y, w \in \mathbb{R}$. Letting $s = y$ and $t = w$, solution set is

$$\left\{ \begin{bmatrix} 2s - t + 3 \\ s \\ 2 - 2t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\},$$

which is a plane in $\mathbb{R}^4$.

2.5.5a Put augmented matrix for system in row-echelon form:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & \lambda & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & \lambda & 4 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & \lambda & 4 \\ 0 & 1 & 2 - \lambda & -1 \\ 0 & 5 & 1 - 2\lambda & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & \lambda & 4 \\ 0 & 1 & 2 - \lambda & -1 \\ 0 & 0 & 3\lambda - 9 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 - \lambda & 2 \\ 0 & 1 & 2 - \lambda & -1 \\ 0 & 0 & 3 - \lambda & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 - \lambda & 0 \end{array}\right].$$

From this it can be seen that there is no value for λ that results in a system having no solution.

2.5.5b If $\lambda \neq 3$ there will be a unique solution. The third equation in the system obtained above is $(3 - \lambda)z = 0$, which gives $z = 0$ when $\lambda \neq 3$. The second equation $y - z = -1$ then yields $y = -1$, and the first equation $x + z = 2$ yields $x = 2$. Thus

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

is the unique solution for *any* $\lambda \neq 3$.

2.5.5c If $\lambda = 3$ there will be infinitely many solutions. The third equation in the system above becomes $0z = 0$ when $\lambda = 3$, and hence z can be any real number. The second equation gives $y = z - 1$, and the first equation gives $x = 2 - z$. Solution set to the system is therefore

$$\left\{ \begin{bmatrix} 2 - z \\ z - 1 \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

2.5.6 Letting $\mathbf{b} = [b_1, b_2, b_3, b_4]^\top$, we have

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ -2 & 3 & -1 & b_2 \\ 3 & -3 & 0 & b_3 \\ 2 & 0 & -2 & b_4 \end{array}\right] \xrightarrow[\substack{-3r_1+r_3\to r_3 \\ -2r_1+r_4\to r_4}]{2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 3 & -3 & b_2 + 2b_1 \\ 0 & -3 & 3 & b_3 - 3b_1 \\ 0 & 0 & 0 & b_4 - 2b_1 \end{array}\right] \xrightarrow{r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 3 & -3 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \\ 0 & 0 & 0 & b_4 - 2b_1 \end{array}\right],$$

so we must have $b_4 - 2b_1 = 0$ and $b_3 + b_2 - b_1 = 0$, which implies $b_4 = 2b_1$ and $b_3 = b_1 - b_2$. We have

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 - b_2 \\ 2b_1 \end{bmatrix}.$$

3.2.2a Let W denote the set. Certainly W contains $\mathbf{O}_2$, the 2×2 matrix with all entries 0, and hence $W \neq \varnothing$. Suppose that $\mathbf{A}, \mathbf{B} \in W$, so that $\mathbf{A}^\top = \mathbf{A}$ and $\mathbf{B}^\top = \mathbf{B}$. Since

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top = \mathbf{A} + \mathbf{B},$$

we have $\mathbf{A} + \mathbf{B} \in W$. Also, for any $c \in \mathbb{R}$,

$$(c\mathbf{A})^\top = c\mathbf{A}^\top = c\mathbf{A},$$

and so $c\mathbf{A} \in W$. Since W is a nonempty set that is closed under vector addition and scalar multiplication, we conclude that it is a subspace of $\mathbb{R}^{2\times 2}$.

3.2.2d Let W denote the set. Observe that

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W,$$

since we need only choose the real numbers $a = 1$ and $b = 0$ to obtain

$$\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand

$$-4\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \notin W,$$

since there is no real number a for which $a^2 = -4$! Therefore W is not a subspace of $\mathbb{R}^{2\times 2}$, since it is not closed under scalar multiplication.

3.4.1 The set $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ does not span $\mathbb{R}^2$ since, for instance, $[1 \;\; 0]^\top$ is not in $\text{Span}(S)$. From

$$a\mathbf{u}_1 + b\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

comes the system

$$\begin{cases} -a + 2b = 1 \\ 3a - 6b = 0 \end{cases}$$

which is readily found to have no solution (multiply the first equation by -3 to see that the system is inconsistent).

3.5.1a Suppose $a, b, c \in \mathbb{R}$ are such that $a\mathbf{u}_1 + b\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$. This gives the system

$$\begin{cases} 2a + 3b - 2c = 0 \\ \qquad\quad b + 3c = 0 \\ -a \qquad\quad + 2c = 0 \end{cases}$$

Solving the system gives $a = b = c = 0$, and therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set of vectors.

3.5.1b Find $a, b, c \in \mathbb{R}$ such that $a\mathbf{u}_1 + b\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{v}$. This gives the system

$$\begin{cases} 2a + 3b - 2c = -6 \\ \qquad\quad b + 3c = -10 \\ -a \qquad\quad + 2c = -5 \end{cases}$$

Solving the system gives $(a, b, c) = (1, -4, -2)$, so

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}.$$

3.5.2 The yz-plane, P_{yz}, is the set of points in $\mathbb{R}^3$ that satisfy the equation $x = 0$. That is,

$$P_{yz} = \left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

From this we see that the set

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

readily shown to be linearly independent, is a basis for P_{yz}.

3.5.3 We have $x = 3z - 2y$, so the plane P consists of points (x, y, z) such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z - 2y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix},$$

where $y, z \in \mathbb{R}$. That is,

$$P = \operatorname{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The vectors

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

are readily verified to be linearly independent, and therefore $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for P.

3.8.1 Suppose $\mathbf{A}$ is $m \times n$, and so has column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$. As a consequence we find that $\mathbf{A}^\top$ has row vectors $\mathbf{a}_1^\top, \ldots, \mathbf{a}_n^\top$, and so since $(\mathbf{a}_k^\top)^\top = \mathbf{a}_k$ for all $1 \leq k \leq n$ by Proposition 2.3(3), by definition we have

$$\mathrm{Col}(\mathbf{A}) = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \mathrm{Span}\{(\mathbf{a}_1^\top)^\top, \ldots, (\mathbf{a}_n^\top)^\top\} = \mathrm{Row}(\mathbf{A}^\top).$$

The proof that $\mathrm{Row}(\mathbf{A}) = \mathrm{Col}(\mathbf{A}^\top)$ is similar.

3.8.2 Since $\mathrm{Col}(\mathbf{A}) = \mathrm{Row}(\mathbf{A}^\top)$ by Problem 3.8.1, we have

$$\mathrm{rank}(\mathbf{A}) = \dim(\mathrm{Col}(\mathbf{A})) = \dim(\mathrm{Row}(\mathbf{A}^\top)) = \mathrm{rank}(\mathbf{A}^\top).$$

3.8.3a Letting $\mathbf{b}_1, \ldots, \mathbf{b}_p$ be the column vectors of $\mathbf{B}$, we have

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \cdots & \mathbf{Ab}_p \end{bmatrix}$$

by Proposition 2.6, which makes clear that $\mathrm{Col}(\mathbf{AB}) = \mathrm{Span}\{\mathbf{Ab}_1, \ldots, \mathbf{Ab}_p\}$. Now, letting $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the column vectors of $\mathbf{A}$, for each $1 \leq k \leq p$ we find by direct computation that

$$\mathbf{Ab}_k = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} = b_{1k}\mathbf{a}_1 + \cdots + b_{nk}\mathbf{a}_n,$$

and so $\mathbf{Ab}_k \in \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Thus

$$\mathrm{Col}(\mathbf{AB}) = \mathrm{Span}\{\mathbf{Ab}_1, \ldots, \mathbf{Ab}_p\} \subseteq \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \mathrm{Col}(\mathbf{A})$$

by the closure properties of the vector space $\mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ (see Proposition 3.30), which shows that $\mathrm{Col}(\mathbf{AB})$ is a subspace of $\mathrm{Col}(\mathbf{A})$, and therefore

$$\mathrm{rank}(\mathbf{AB}) = \dim(\mathrm{Col}(\mathbf{AB})) \leq \dim(\mathrm{Col}(\mathbf{A})) = \mathrm{rank}(\mathbf{A})$$

by Theorem 3.56(2).

3.8.3b We use Problems 3.8.2 and 3.8.3a to do this:

$$\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}((\mathbf{AB})^\top) = \mathrm{rank}(\mathbf{B}^\top\mathbf{A}^\top) \leq \mathrm{rank}(\mathbf{B}^\top) = \mathrm{rank}(\mathbf{B}).$$

4.2.1 Suppose that $\mathbf{x}_0$ is a solution to the system $\mathbf{Ax} = \mathbf{b}$, so that $\mathbf{Ax}_0 = \mathbf{b}$. Let S represent the solution set of the system. We must prove that $S = \mathbf{x}_0 + \mathrm{Nul}(L)$.

Let $\mathbf{z} \in \mathbf{x}_0 + \mathrm{Nul}(L)$. Then $\mathbf{z} = \mathbf{x}_0 + \mathbf{y}$ for some $\mathbf{y} \in \mathrm{Nul}(L)$, and so

$$\mathbf{Az} = \mathbf{A}(\mathbf{x}_0 + \mathbf{y}) = \mathbf{Ax}_0 + \mathbf{Ay} = \mathbf{b} + L(\mathbf{y}) = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

This shows that $\mathbf{z}$ satisfies the system. That is, $\mathbf{z} \in S$ and we conclude that $\mathbf{x}_0 + \mathrm{Nul}(L) \subseteq S$.

Next, let $\mathbf{z} \in S$, which implies that $\mathbf{Az} = \mathbf{b}$. Observing that $\mathbf{z} = \mathbf{x}_0 + (\mathbf{z} - \mathbf{x}_0)$, where

$$L(\mathbf{z} - \mathbf{x}_0) = \mathbf{A}(\mathbf{z} - \mathbf{x}_0) = \mathbf{Az} - \mathbf{Ax}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

shows $\mathbf{z} - \mathbf{x}_0 \in \mathrm{Nul}(L)$, we see $\mathbf{z} \in \mathbf{x}_0 + \mathrm{Nul}(L)$ and conclude that $S \subseteq \mathbf{x}_0 + \mathrm{Nul}(L)$.

Therefore $S = \mathbf{x}_0 + \mathrm{Nul}(L)$.

4.3.1 We have $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$, and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Now,

$$L(\mathbf{v}_1) = L\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-5(3)+13(1)\\-7(3)+16(1)\end{bmatrix} = \begin{bmatrix}1\\-2\\-5\end{bmatrix}.$$

We need the $\mathcal{C}$-coordinates of $L(\mathbf{v}_1)$, which means finding a_1, a_2, a_3 such that

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + a_2\begin{bmatrix}-1\\2\\2\end{bmatrix} + a_3\begin{bmatrix}0\\1\\2\end{bmatrix} = L(\mathbf{v}_1);$$

that is,

$$\begin{cases} a_1 - a_2 = 1 \\ 2a_2 + a_3 = -2 \\ -a_1 + 2a_2 + 2a_3 = -5 \end{cases}$$

which solves to give $a_1 = 1$, $a_2 = 0$, and $a_3 = -2$. Thus

$$[L(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix}1\\0\\-2\end{bmatrix}.$$

Next,

$$L(\mathbf{v}_2) = L\left(\begin{bmatrix}5\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-3\end{bmatrix}.$$

We need the $\mathcal{C}$-coordinates of $L(\mathbf{v}_2)$, so we find a_1, a_2, a_3 such that

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + a_2\begin{bmatrix}-1\\2\\2\end{bmatrix} + a_3\begin{bmatrix}0\\1\\2\end{bmatrix} = L(\mathbf{v}_2).$$

Like before, this yields a system of equations. We put its augmented matrix into row-echelon form:

$$\left[\begin{array}{ccc|c}1&-1&0&2\\0&2&1&1\\-1&2&2&-3\end{array}\right] \sim \left[\begin{array}{ccc|c}1&-1&0&2\\0&2&1&1\\0&1&2&-1\end{array}\right] \sim \left[\begin{array}{ccc|c}1&-1&0&2\\0&1&2&-1\\0&2&1&1\end{array}\right] \sim \left[\begin{array}{ccc|c}1&-1&0&2\\0&1&2&-1\\0&0&-3&3\end{array}\right]$$

which solves to give $a_1 = 3$, $a_2 = 1$, and $a_3 = -1$. Thus

$$[L(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix}3\\1\\-1\end{bmatrix}.$$

The $\mathcal{BC}$-matrix of L is therefore

$$[L]_{\mathcal{BC}} = \Big[[L(\mathbf{v}_1)]_{\mathcal{C}} \;\; [L(\mathbf{v}_2)]_{\mathcal{C}}\Big] = \begin{bmatrix}1&3\\0&1\\-2&-1\end{bmatrix}.$$

4.4.1a Letting $\mathbf{e}_1 = [1,0]^\top$ and $\mathbf{e}_2 = [0,1]^\top$, by Theorem 4.27 we have

$$\mathbf{I}_{\mathcal{E}\mathcal{B}} = \big[[\mathbf{e}_1]_\mathcal{B} \quad [\mathbf{e}_2]_\mathcal{B}\big],$$

and so we must find the $\mathcal{B}$-coordinates of $\mathbf{e}_1$ and $\mathbf{e}_2$. Letting $\mathbf{b}_1 = [1,2]^\top$ and $\mathbf{b}_2 = [-2,1]^\top$, so that $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$, we must find $x_1, x_2, y_1, y_2 \in \mathbb{R}$ so $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = \mathbf{e}_1$ and $y_1\mathbf{b}_1 + y_2\mathbf{b}_2 = \mathbf{e}_2$; that is,

$$\begin{cases} x_1 - 2x_2 = 1 \\ 2x_1 + \ \ x_2 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} y_1 - 2y_2 = 0 \\ 2y_1 + \ \ y_2 = 1 \end{cases}$$

Solving these systems gives $(x_1, x_2) = (\frac{1}{5}, -\frac{2}{5})$ and $(y_1, y_2) = (\frac{2}{5}, \frac{1}{5})$. Thus $[\mathbf{e}_1]_\mathcal{B} = [\frac{1}{5}, -\frac{2}{5}]^\top$ and $[\mathbf{e}_2]_\mathcal{B} = [\frac{2}{5}, \frac{1}{5}]^\top$, and we obtain

$$\mathbf{I}_{\mathcal{E}\mathcal{B}} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

4.4.1b We have

$$[\mathbf{x}]_\mathcal{B} = \mathbf{I}_{\mathcal{E}\mathcal{B}}[\mathbf{x}]_\mathcal{E} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{8}{5} \\ -\frac{9}{5} \end{bmatrix}.$$

5.3.1 Letting

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 4 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

the system of equations becomes the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. We evaluate $\det(\mathbf{A})$ by expanding along the first column:

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & 4 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -12 - 2(-5) = -2.$$

Now, letting $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$, we have

$$\det(\mathbf{b} \ \mathbf{a}_2 \ \mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & 3 & 1 \end{vmatrix} = 10, \quad \det(\mathbf{a}_1 \ \mathbf{b} \ \mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 3 & 1 \end{vmatrix} = 0,$$

and

$$\det(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{vmatrix} = -6.$$

By Cramer's Rule we have

$$x = \frac{\det(\mathbf{b} \ \mathbf{a}_2 \ \mathbf{a}_3)}{\det(\mathbf{A})} = -5, \quad y = \frac{\det(\mathbf{a}_1 \ \mathbf{b} \ \mathbf{a}_3)}{\det(\mathbf{A})} = 0, \quad z = \frac{\det(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b})}{\det(\mathbf{A})} = 3,$$

and so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$$

is the solution to the system.

6.2.1a (i) Characteristic equation: $t^2 - 2t - 3 = 0$. (ii) Eigenvalues: $3, -1$. (iii) Basis for eigenspace corresponding to

$$\lambda = 3 : \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}; \qquad \lambda = -1 : \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

6.2.1b (i) Characteristic equation: $t^2 - 12 = 0$. (ii) Eigenvalues: $\sqrt{12}, -\sqrt{12}$. (iii) Basis for eigenspace corresponding to

$$\lambda = \sqrt{12} : \left\{ \begin{bmatrix} 3 \\ \sqrt{12} \end{bmatrix} \right\}; \qquad \lambda = -\sqrt{12} : \left\{ \begin{bmatrix} -3 \\ \sqrt{12} \end{bmatrix} \right\}.$$

6.2.1c (i) Characteristic equation: $t^2 + 3 = 0$. (ii) Eigenvalues: no real eigenvalues. (iii) Eigenspaces: none corresponding to real eigenvalues.

6.2.1d (i) Characteristic equation: $t^2 = 0$. (ii) Eigenvalue: 0. (iii) Basis for eigenspace corresponding to $\lambda = 0$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

6.2.2a (i) Characteristic equation: $t^3 - 6t^2 + 11t - 6 = 0$. (ii) Eigenvalues: $\lambda = 1, 2, 3$. (iii) Basis for eigenspace corresponding to

$$\lambda = 1 : \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}; \qquad \lambda = 2 : \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}; \qquad \lambda = 3 : \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

6.2.2b (i) Characteristic equation: $t^3 - 2t = 0$. (ii) Eigenvalues: $0, \sqrt{2}, -\sqrt{2}$. (iii) Basis for eigenspace corresponding to

$$\lambda = 0 : \left\{ \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \right\}; \qquad \lambda = \sqrt{2} : \left\{ \begin{bmatrix} 15 + 5\sqrt{2} \\ -1 + 2\sqrt{2} \\ 7 \end{bmatrix} \right\}; \qquad \lambda = -\sqrt{2} : \left\{ \begin{bmatrix} 15 - 5\sqrt{2} \\ -1 - 2\sqrt{2} \\ 7 \end{bmatrix} \right\}.$$

6.2.2c (i) Characteristic equation: $t^3 - 2t^2 - 15t + 36 = 0$. (ii) Eigenvalues: $-4, 3$. (iii) Basis for eigenspace corresponding to

$$\lambda = -4 : \left\{ \begin{bmatrix} -6 \\ 8 \\ 3 \end{bmatrix} \right\}; \qquad \lambda = 3 : \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

6.2.3a (i) Characteristic equation: $(t-1)^2(t+2)(t+1)=0$. (ii) Eigenvalues: $1, -2, -1$. (iii) Basis for eigenspace corresponding to

$$\lambda = 1: \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}; \qquad \lambda = -2: \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}; \qquad \lambda = -1: \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

6.2.3b (i) Characteristic equation: $(t-4)^2(t^2+3)=0$. (ii) Real eigenvalue: 4. (iii) Basis for eigenspace corresponding to $\lambda = 4$:

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

6.6.1a Characteristic polynomial is

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \begin{vmatrix} 3-t & 2 \\ 2 & 3-t \end{vmatrix} = (3-t)^2 - 4 = (t-5)(t-1),$$

and so the eigenvalues of $\mathbf{A}$ are 1, 5.

6.6.1b For the eigenvalue 1 the associated eigenspace is the solution set for $\mathbf{Ax} = \mathbf{x}$, where

$$\mathbf{Ax} = \mathbf{x} \quad \Rightarrow \quad (\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields $x_2 = -x_1$. Hence

$$E_{\mathbf{A}}(1) = \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

For the eigenvalue 5 the associated eigenspace is the solution set for $\mathbf{Ax} = 5\mathbf{x}$, where

$$\mathbf{Ax} = 5\mathbf{x} \quad \Rightarrow \quad (\mathbf{A} - 5\mathbf{I})\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields $x_2 = x_1$. Hence

$$E_{\mathbf{A}}(5) = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

6.6.1c Let

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

It is routine to verify that

$$\mathbf{PDP}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \mathbf{A}.$$

6.6.1d We have

$$\mathbf{A}^{50} = (\mathbf{PDP}^{-1})^{50} = \mathbf{PD}^{50}\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+5^{50} & -1+5^{50} \\ -1+5^{50} & 1+5^{50} \end{bmatrix}.$$

Next, let

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix},$$

and note that $(\mathbf{PCP}^{-1})^2 = \mathbf{PC}^2\mathbf{P}^{-1} = \mathbf{PDP}^{-1} = \mathbf{A}$. Hence

$$\mathbf{A}^{1/2} = \mathbf{PCP}^{-1} = \frac{1}{2}\begin{bmatrix} 1+\sqrt{5} & -1+\sqrt{5} \\ -1+\sqrt{5} & 1+\sqrt{5} \end{bmatrix}.$$

6.6.2 The matrix $\mathbf{A}$ is diagonalizable, with

$$\mathbf{P} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

6.6.3 The matrix $\mathbf{A}$ is diagonalizable, with

$$\mathbf{P} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix}.$$

6.6.4 Find the characteristic polynomial:

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \begin{vmatrix} 2-t & 0 & -2 \\ 0 & 3-t & 0 \\ 0 & 0 & 3-t \end{vmatrix} = (2-t)\begin{vmatrix} 3-t & 0 \\ 0 & 3-t \end{vmatrix} = (2-t)(3-t)^2.$$

The characteristic equation is $(2-t)(3-t)^2 = 0$, which has solution set $\{2, 3\}$. Hence the eigenvalues of $\mathbf{A}$ are 2 and 3.

The eigenspace corresponding to 2 is

$$E_{\mathbf{A}}(2) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{Ax} = 2\mathbf{x}\} = \{\mathbf{x} : (\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}\}.$$

Passing to the augmented matrix for the system $(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$, we have

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus the solution set of the system is

$$\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : y = z = 0 \text{ and } x \in \mathbb{R}\right\} = \left\{\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

A basis for $E_{\mathbf{A}}(2)$ is thus $\mathcal{B}_1 = \{[1, 0, 0]^\top\}$.

The eigenspace corresponding to 3 is

$$E_{\mathbf{A}}(3) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = 3\mathbf{x}\} = \{\mathbf{x} : (\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}\}.$$

Passing to the augmented matrix for the system $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$, we have

$$\left[\begin{array}{rrr|r} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the solution set of the system is

$$\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -2z \text{ and } y, z \in \mathbb{R}\right\} = \left\{ s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

A basis for $E_{\mathbf{A}}(3)$ is thus $\mathcal{B}_2 = \left\{[0, 1, 0]^\top, [-2, 0, 1]^\top\right\}$.

A spectral basis for $\mathbf{A}$ (i.e. a basis for $\mathbb{R}^3$ consisting of linearly independent eigenvectors of $\mathbf{A}$) is the ordered basis

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right).$$

The eigenvalues corresponding to these eigenvalues are 2, 3, and 3, respectively. Therefore the diagonal matrix we seek is

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

As for $\mathbf{P}$, that is the 3×3 matrix with column vectors being the vectors in $\mathcal{B}$ in the order that they appear:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7.3.1a $\left\{\frac{1}{\sqrt{10}}[1, -3]^\top,\ \frac{1}{\sqrt{10}}[3, 1]^\top\right\}$

7.3.1b $\left\{[1, 0]^\top,\ [0, -1]^\top\right\}$

7.3.2a $\left\{\frac{1}{\sqrt{3}}[1, 1, 1]^\top,\ \frac{1}{\sqrt{2}}[-1, 1, 0]^\top,\ \frac{1}{\sqrt{6}}[1, 1, -2]^\top\right\}$

7.3.2b $\left\{[1, 0, 0]^\top,\ \frac{1}{\sqrt{53}}[0, 7, -2]^\top,\ \frac{1}{\sqrt{53}}[0, 2, 7]^\top\right\}$

7.3.3 $\left\{\frac{1}{\sqrt{5}}[0, 2, 1, 0]^\top,\ \frac{1}{\sqrt{30}}[5, -1, 2, 0]^\top,\ \frac{1}{\sqrt{10}}[1, 1, -2, -2]^\top,\ \frac{1}{\sqrt{15}}[1, 1, -2, 3]^\top\right\}$

7.3.4a By Ye Olde Gram-Schmidt Process,

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix},$$

and

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

An orthogonal basis for W is therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\},$$

the latter basis obtained by replacing $\mathbf{w}_2$ with $2\mathbf{w}_2$ to rid ourselves of fractions.

7.3.4b Find the norms of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ found above:

$$\|\mathbf{w}_1\| = \sqrt{2}, \quad \|\mathbf{w}_2\| = \frac{1}{\sqrt{2}}, \quad \|\mathbf{w}_3\| = \sqrt{10}.$$

An orthonormal basis for W is thus

$$\left\{ \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} \right\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

7.3.5b $\{[1,-1,-1,1,1]^\top,\ [3,0,3,-3,3]^\top,\ [2,0,2,2,-2]^\top\}$.