MATH 250 QUIZ #5 (FALL 2020)

1 Use the Laplace transform method to solve the initial-value problem

$$y' + y = f(t), \quad y(0) = 0,$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1 \end{cases}$$

First we determine that $f(t) = 1 - 2\mathfrak{u}(t-1)$, so

$$y' + y = 1 - 2\mathfrak{u}(t-1),$$

and thus, taking the Laplace transform of each side (and letting $Y(s) = \mathcal{L}[y](s)$), we have

$$\mathcal{L}[y' + y] = \mathcal{L}[1 - 2\mathfrak{u}(t-1)] \;\Rightarrow\; sY - y(0) + Y = \frac{1}{s} - \frac{2e^{-s}}{s} \;\Rightarrow\; Y = \frac{1}{s(s+1)} - \frac{2e^{-s}}{s(s+1)}.$$

Now, since

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

we find that

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] - 2\mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s+1)}\right] = (1 - e^{-t}) - 2\mathcal{L}^{-1}\left[e^{-s}\mathcal{L}[g(t+1)]\right],$$

where

$$\mathcal{L}[g(t+1)] = \frac{1}{s(s+1)}$$

implies $g(t+1) = 1 - e^{-t}$. Using the property $\mathcal{L}[g(t)u(t-a)] = e^{-as}\mathcal{L}[g(t+a)]$, we finally obtain

$$y(t) = (1 - e^{-t}) - 2(1 - e^{1-t})\mathfrak{u}(t-1).$$