Math 250 Quiz #4 (Fall 2020)

1 A $\frac{1}{4}$ kg mass is attached to a spring with stiffness 8 N/m. The damping constant is $\frac{1}{4}$ N-sec/m. If the mass is moved 1 m to the left of equilibrium and released, what is the equation of motion, and what is the maximum displacement to the right the mass will attain?

The initial value problem is

$$\tfrac{1}{4}y'' + \tfrac{1}{4}y' + 8y = 0, \quad y(0) = -1, \quad y'(0) = 0.$$

The auxiliary equation can we written as $r^2 + r + 32$, which has solutions $r = -\frac{1}{2} \pm \frac{\sqrt{127}}{2}i$, and so the general solution to the differential equation is

$$y(t) = e^{-t/2}\left(c_1 \cos \tfrac{\sqrt{127}}{2}t + c_2 \sin \tfrac{\sqrt{127}}{2}t\right).$$

Using $y(0) = -1$ we get $c_1 = -1$, and we then find that

$$y'(t) = -\tfrac{1}{2}e^{-t/2}\left(c_2 \sin \tfrac{\sqrt{127}}{2}t - \cos \tfrac{\sqrt{127}}{2}t\right) + e^{-t/2}\left(\tfrac{\sqrt{127}}{2}c_2 \cos \tfrac{\sqrt{127}}{2}t + \tfrac{\sqrt{127}}{2} \sin \tfrac{\sqrt{127}}{2}t\right). \qquad (1)$$

Using this and the initial condition $y'(0) = 0$, we find that $c_2 = -\frac{1}{\sqrt{127}}$, and thus the equation of motion is

$$y(t) = -e^{-t/2}\left(\cos \tfrac{\sqrt{127}}{2}t + \tfrac{1}{\sqrt{127}} \sin \tfrac{\sqrt{127}}{2}t\right). \qquad (2)$$

Maximum displacement to the right will occur at the first time $t > 0$ for which $y'(t) = 0$. Using equation (1), $y'(t) = 0$ implies that

$$\sin\left(\tfrac{\sqrt{127}}{2}t\right) = 0,$$

which has as its smallest solution on $(0, \infty)$ the value $t = \frac{2\pi}{\sqrt{127}}$. At this time the displacement is, using (2),

$$y\left(\tfrac{2\pi}{\sqrt{127}}\right) = e^{-\pi/\sqrt{127}} \approx 0.76 \text{ m}$$

2 Find the interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{5^{2n}} x^{2n+1}.$$

Using the Ratio Test, we find that

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}x^{2n+3}}{5^{2n+2}} \cdot \frac{5^{2n}}{(-1)^{n+1}x^{2n+1}}\right| = \lim_{n\to\infty} \frac{x^2}{25} = \frac{x^2}{25}.$$

The test concludes the series converges for any x such that $x^2/25 < 1$, or $x \in (-5, 5)$. If $x^2/25 = 1$ (i.e. $x = \pm 5$) the test is inconclusive.

If $x = -5$ the series becomes $\sum 5(-1)^{3n+2}$, but since $\lim_{n\to\infty} 5(-1)^{3n+2} \neq 0$ the Divergence Test concludes that the series diverges. If $x = 5$ the series becomes $\sum 5(-1)^{n+1}$, which again diverges by the Divergence Test. Therefore $(-5, 5)$ is the interval of convergence.

3 Write

$$\sum_{n=2}^{\infty} n c_n x^{n-2} + 6 \sum_{n=0}^{\infty} c_n x^{n+2}.$$

as a single series.

Reindex to get x^n in both series:

$$\sum_{n=0}^{\infty} n c_{n+2} x^n + \sum_{n=2}^{\infty} 6 c_{n-2} x^n.$$

Throw the first two terms out of the first series and add to get

$$2c_2 + 3c_3 x + \sum_{n=2}^{\infty} \big[(n+2)c_{n+2} + 6c_{n-2}\big] x^n.$$