Math 250
Summer 2024
Exam 2

Name:

1. 10 pts. Solve $(y^2 + 2xy)dx - x^2 dy = 0$ by finding an integrating factor of the form

$$\exp\left[\int\left(\frac{\partial M/dy - \partial N/dx}{N}\right)dx\right] \quad \text{or} \quad \exp\left[\int\left(\frac{\partial N/dx - \partial M/dy}{M}\right)dy\right].$$

2. 10 pts. Solve the differential equation $x^2 y'' + (y')^2 = 0$ using the reduction of order method.

3. 10 pts. Solve the initial-value problem using the reduction of order method:

$$y'' + yy' = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

4. 10 pts. each Find the general solution to the differential equation.
   (a) $y'' + 4y' - y = 0$
   (b) $2y'' - 3y' + 4y = 0$

5. 15 pts. Solve the initial-value problem by the method of undetermined coefficients:

$$y'' + y' + y = \sin x, \quad y(1) = 1, \quad y'(1) = 1.$$

6. 15 pts. Use the method of variation of parameters to find a particular solution to

$$y'' + 2y' + 5y = e^{-x} \sec 2x,$$

then find the general solution.

7. 10 pts. One solution to $xy'' - (2x+1)y' + (x+1)y = 0$ is $y_1(x) = e^x$. Find the general solution to the differential equation.

Method of Undetermined Coefficients. Let $P_m(x)$ be a nonzero polynomial of degree m, and let $y_p(x)$ denote a particular solution to $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$.

1. If $f(x) = P_m(x)e^{\alpha x}$, then
$$y_p(x) = x^s e^{\alpha x} \sum_{k=0}^{m} A_k x^k,$$
where $s = 0$ if α is not a root of the auxiliary equation, otherwise s equals the multiplicity of α as a root of the auxiliary equation.
2. If $f(x) = P_m(x)e^{\alpha x}\cos\beta x$ or $f(x) = P_m(x)e^{\alpha x}\sin\beta x$ for $\beta \neq 0$, then
$$y_p(x) = x^s e^{\alpha x}\left(\cos\beta x \sum_{k=0}^{m} A_k x^k + \sin\beta x \sum_{k=0}^{m} B_k x^k\right),$$
where $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation, otherwise s equals the multiplicity of $\alpha + i\beta$ as a root of the auxiliary equation.

Method of Variation of Parameters (2nd-Order Case).

For $y'' + p(x)y' + q(x) = f(x)$,

$$y_p(x) = y_1(x)\int \frac{-f(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}dx + y_2(x)\int \frac{f(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}dx$$

Some Most Excellent Formulae.

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, for $a \in (0, \infty)$
2. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, for $a \neq 0$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$, for $a \in (0, \infty)$
4. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$
5. $\displaystyle\int \cot x\,dx = \ln|\sin x| + c$
6. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + c$
7. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + c$