Math 250
Summer 2017 — Name:
Exam 2

1. $\boxed{\text{10 pts.}}$ Given that $y_1(t) = t^2$ is a solution to

$$t^2y'' + 2ty' - 6y = 0,$$

use reduction of order to find a second solution $y_2(t)$.

2. $\boxed{\text{10 pts. each}}$ Find the general solution to each.

   (a) $y'' + 8y' + 16y = 0$

   (b) $3y''' + 10y'' + 15y' + 4y = 0$

3. $\boxed{\text{10 pts.}}$ Solve the initial-value problem:

$$y'' - 6y' + 25y = 0, \quad y(0) = 3, \quad y'(0) = 1.$$

4. $\boxed{\text{10 pts.}}$ Find a homogeneous linear differential equation with constant coefficients whose general solution is

$$y = c_1 + c_2e^{2x}\cos 5x + c_3e^{2x}\sin 5x.$$

5. $\boxed{\text{10 pts.}}$ Given that $y_1 = x^4$ is a solution to

$$x^2y'' - 7xy' + 16y = 0,$$

use reduction of order to find a second solution y_2 that is linearly independent from y_1.

6. Consider the equation $y''' - 3y'' + 3y' - y = -4e^x$.

   (a) $\boxed{\text{10 pts.}}$ Use the Method of Undetermined Coefficients to find a particular solution to the equation.

   (b) $\boxed{\text{5 pts.}}$ Give the general solution to the equation.

7. Consider the equation $\frac{1}{4}y'' + y' + y = x^2 - 2x$.

   (a) $\boxed{\text{15 pts.}}$ Use the Method of Undetermined Coefficients to find a particular solution to the equation.

   (b) $\boxed{\text{5 pts.}}$ Give the general solution to the equation.

8. $\boxed{\text{20 pts.}}$ Use the Method of Variation of Parameters to find a particular solution to

$$y'' - 2y' + 2y = e^x \tan x,$$

and then find a general solution.

9. $\boxed{\text{10 pts.}}$ Solve the nonlinear initial-value problem:

$$y'y'' = 4x, \quad y(1) = 5, \quad y'(1) = 2.$$

10. $\boxed{\text{10 pts.}}$ Find the first six nonzero terms of a Taylor series solution, centered at 0, of the initial-value problem:

$$y'' + y^2 = 1, \quad y(0) = 2, \quad y'(0) = 3$$

11. A 1-kilogram mass is attached to a spring whose constant is 16 N/m, and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the velocity.

    (a) $\boxed{\text{15 pts.}}$ Find the equation of motion of the mass if it is initially released 1 meter below the equilibrium position with an upward velocity of 12 m/s.

    (b) $\boxed{\text{5 pts.}}$ Find the first time when the mass passes through the equilibrium position.

Method of Undetermined Coefficients. Let $P_m(x)$ be a nonzero polynomial of degree m, and let $y_p(x)$ denote a particular solution to $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$.

1. If $f(x) = P_m(x)e^{\alpha x}$, then
$$y_p(x) = x^s e^{\alpha x} \sum_{k=0}^{m} A_k x^k,$$
where $s = 0$ if α is not a root of the auxiliary equation, otherwise s equals the multiplicity of α as a root of the auxiliary equation.

2. If $f(x) = P_m(x)e^{\alpha x}\cos\beta x$ or $f(x) = P_m(x)e^{\alpha x}\sin\beta x$ for $\beta \neq 0$, then
$$y_p(x) = x^s e^{\alpha x}\left(\cos\beta x \sum_{k=0}^{m} A_k x^k + \sin\beta x \sum_{k=0}^{m} B_k x^k\right),$$
where $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation, otherwise s equals the multiplicity of $\alpha + i\beta$ as a root of the auxiliary equation.

Method of Variation of Parameters (2nd-Order Case).

For $y'' + p(x)y' + q(x) = f(x)$,
$$u_1(x) = \int \frac{-y_2(x)f(x)}{W[y_1, y_2](x)}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)f(x)}{W[y_1, y_2](x)}\,dx$$

Some Most Excellent Formulae.

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, for $a \in (0, \infty)$

2. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, for $a \neq 0$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$, for $a \in (0, \infty)$

4. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$

5. $\displaystyle\int \cot x\,dx = \ln|\sin x| + c$

6. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + c$

7. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + c$