Math 250
Summer 2016 Name:
Exam 2

1. $\boxed{\text{15 pts.}}$ A thermometer reading 70°F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer notes that the thermometer reads 120°F after half a minute and 160°F after one minute. How hot is the oven?

2. $\boxed{\text{15 pts.}}$ A large tank is partially filled with 400 liters of water in which 5 kilograms of sugar is dissolved. Water containing 0.05 kg of sugar per liter is pumped into the tank at a rate of 20 L/min. The well-mixed solution is meanwhile pumped out at a slower rate of 15 L/min. Find the number of kilograms of sugar in the tank after one hour.

3. $\boxed{\text{10 pts.}}$ Using either the Wronskian determinant or the definition of linear independence, determine whether the functions $f(x) = x^2$, $g(x) = 6x^2 - 1$, and $h(x) = 2x^2 + 3$ are linearly independent on $(-\infty, \infty)$.

4. $\boxed{\text{10 pts.}}$ Find the general solution to $y'' - 10y' + 25y = 0$.

5. $\boxed{\text{10 pts.}}$ Find the general solution to $y^{(4)} + y''' + y'' = 0$.

6. $\boxed{\text{10 pts.}}$ Given that $y_1(t) = t^2$ is a solution to

$$t^2y'' + 2ty' - 6y = 0,$$

use reduction of order to find a second solution $y_2(t)$.

7. Consider the equation $y'' + 2y' = 2t + 5 - e^{-2t}$.

   (a) $\boxed{\text{15 pts.}}$ Use the Method of Undetermined Coefficients and the Superposition Principle to find a particular solution to the equation.

   (b) $\boxed{\text{5 pts.}}$ Give the general solution to the equation.

8. $\boxed{\text{10 pts.}}$ Use the Method of Undetermined Coefficients to solve the initial-value problem

$$y'' + 4y = -2, \quad y(\pi/8) = \tfrac{1}{2}, \quad y'(\pi/8) = 2.$$

9. $\boxed{\text{20 pts.}}$ Use the Method of Variation of Parameters to find a particular solution to

$$y'' + y = \sec^2 t,$$

and then find a general solution.

10. $\boxed{\text{10 pts.}}$ A 1-kilogram mass is attached to a spring whose constant is 16 N/m, and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Determine the equation of motion if the mass is initially released from a point 1 meter below the equilibrium position with an upward velocity of 12 m/s.

Method of Undetermined Coefficients. Let $P_m(t)$ be a nonzero polynomial of degree m, and let $y_p(t)$ denote a particular solution to $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(t)$.

1. If $f(t) = P_m(t)e^{\alpha t}$, then
$$y_p(t) = t^s e^{\alpha t} \sum_{k=0}^{m} A_k t^k,$$
where $s = 0$ if α is not a root of the auxiliary equation, otherwise s equals the multiplicity of α as a root of the auxiliary equation.

2. If $f(t) = P_m(t)e^{\alpha t}\cos\beta t$ or $f(t) = P_m(t)e^{\alpha t}\sin\beta t$ for $\beta \neq 0$, then
$$y_p(t) = t^s e^{\alpha t}\cos\beta t \sum_{k=0}^{m} A_k t^k + t^s e^{\alpha t}\sin\beta t \sum_{k=0}^{m} B_k t^k,$$
where $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation, otherwise s equals the multiplicity of $\alpha + i\beta$ as a root of the auxiliary equation.

Reduction of Order.

$$y_2(t) = y_1(t)\int \frac{e^{-\int P(t)dt}}{y_1^2(t)}dt.$$

Method of Variation of Parameters (2nd-Order Case).

$$v_1(t) = \frac{1}{a_2}\int \frac{-y_2(t)f(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}dt \quad \text{and} \quad v_2(t) = \frac{1}{a_2}\int \frac{y_1(t)f(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}dt.$$

Some Most Excellent Formulae.

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, for $a \in (0,\infty)$
2. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, for $a \neq 0$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$, for $a \in (0,\infty)$
4. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$
5. $\displaystyle\int \cot x\,dx = \ln|\sin x| + c$
6. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + c$
7. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + c$