Math 250
Spring 2018
Exam 3

Name:

1. [15 pts.] Solve the nonlinear ODE

$$xy'' = y' + x(y')^2.$$

Assume the first arbitrary constant that arises in your work is positively valued. (Hint: recall the Bernoulli equations.)

2. Consider the equation $y'' + 3y = -48x^2e^{3x}$.

   (a) [10 pts.] Use the Method of Undetermined Coefficients to find a particular solution to the equation.

   (b) [5 pts.] Give the general solution to the equation.

3. [20 pts.] Use the Method of Undetermined Coefficients to solve the IVP

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t, \quad x(0) = 0, \quad x'(0) = 0,$$

assuming $\omega > 0$.

4. [20 pts.] Use the Method of Variation of Parameters to find a particular solution to

$$y'' - 2y' + 2y = e^x \tan x,$$

and then find a general solution.

Method of Undetermined Coefficients. Let $P_m(x)$ be a nonzero polynomial of degree m, and let $y_p(x)$ denote a particular solution to $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)$.

1. If $f(x) = P_m(x)e^{\alpha x}$, then
$$y_p(x) = x^s e^{\alpha x} \sum_{k=0}^{m} A_k x^k,$$
where $s = 0$ if α is not a root of the auxiliary equation, otherwise s equals the multiplicity of α as a root of the auxiliary equation.
2. If $f(x) = P_m(x)e^{\alpha x}\cos\beta x$ or $f(x) = P_m(x)e^{\alpha x}\sin\beta x$ for $\beta \neq 0$, then
$$y_p(x) = x^s e^{\alpha x}\left(\cos\beta x \sum_{k=0}^{m} A_k x^k + \sin\beta x \sum_{k=0}^{m} B_k x^k\right),$$
where $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation, otherwise s equals the multiplicity of $\alpha + i\beta$ as a root of the auxiliary equation.

Method of Variation of Parameters (2nd-Order Case).

$$u_1(x) = \frac{1}{a_2}\int \frac{-y_2(x)f(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}dx \quad \text{and} \quad u_2(x) = \frac{1}{a_2}\int \frac{y_1(x)f(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}dx$$

Some Most Excellent Formulae.

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, for $a \in (0, \infty)$
2. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, for $a \neq 0$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$, for $a \in (0, \infty)$
4. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$
5. $\displaystyle\int \cot x\,dx = \ln|\sin x| + c$
6. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + c$
7. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + c$