Math 250 Exam #3 Key (Summer 2024)

1 Auxiliary equation $r^3 - 3r^2 + 4r - 2 = 0$ has roots 1 and $1 \pm i$. General solution:

$$y = (c_1 + c_2 \cos x + c_3 \sin x)e^x.$$

2 Auxiliary equation $r^3 - r = 0$ has roots $-1, 0, 1$. General solution is $y = c_1 + c_2 e^x + c_3 e^{-x}$. Using the initial conditions, we find that $c_1 + c_2 + c_3 = 4$, $c_2 - c_3 = 4$, and $c_2 + c_3 = 4$. Solving this system of equations yields $c_1 = 0$, $c_2 = 4$, $c_3 = 0$. Solution to IVP: $y = 4e^x$.

3 Model: $50x'' + 200x = 0$, $x(0) = 0$, $x'(0) = -10$. Here $x(t) < 0$ is the position that compresses the (vertically hanging) spring. From the ODE comes

$$x(t) = c_1 \cos 2t + c_2 \sin 2t,$$

and with the initial conditions we find $x(t) = -5 \sin 2t$. Period of motion is π seconds. Now find t such that $x'(t) = 5$ m/s, or $\cos 2t = -\frac{1}{2}$. The times are

$$t = \frac{\pi}{3} + \pi n \quad \text{and} \quad t = \frac{2\pi}{3} + \pi n,$$

$n \geq 0$ an integer.

4 For $|x| < 1$,

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n \quad \hookrightarrow \quad -\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \quad \hookrightarrow \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} -nx^{n-1}.$$

5 Let $y = \sum_{k=0}^{\infty} c_k x^k$, so $y' = \sum_{k=1}^{\infty} kc_k x^{k-1}$. DE then becomes

$$\sum_{k=1}^{\infty} kc_k x^{k-1} - \sum_{k=0}^{\infty} c_k x^k = x^2 \quad \hookrightarrow \quad \sum_{k=0}^{\infty} [(k+1)c_{k+1} - c_k]x^k = x^2.$$

Thus we have

$$(c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + \sum_{k=0}^{3} [(k+1)c_{k+1} - c_k]x^k = x^2,$$

so that $c_1 - c_0 = 0$, $2c_2 - c_1 = 0$, $3c_3 - c_2 = 1$, and $(k+1)c_{k+1} - c_k = 0$ for $k \geq 3$. We only need to find a particular solution to the DE, so let $c_0 = 0$. Then $c_1 = c_2 = 0$, $c_3 = \frac{1}{3}$, and $c_{k+1} = \frac{1}{k+1}c_k$ for $k \geq 3$. Using the recurrence relation, we have $c_4 = \frac{1}{12} = \frac{2}{4!}$, $c_5 = \frac{2}{5!}$, and generally $c_k = \frac{2}{k!}$ for $k \geq 3$. We get

$$y = \sum_{k=3}^{\infty} \frac{2}{k!} x^k = 2\sum_{k=0}^{\infty} \frac{x^k}{k!} - 2 - 2x - x^2 = 2e^x - 2 - 2x - x^2.$$

6 Put $y = \sum_{n=0}^{\infty} c_n x^n$ into the ODE to get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0,$$

so

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1)c_{n+2} + nc_n + 2c_n\big]x^n = 0,$$

and hence $(n+2)(n+1)c_{n+2} + nc_n + 2c_n = 0$ for all $n \geq 0$. Solve for c_{n+2}:

$$c_{n+2} = -\frac{c_n}{n+1}, \quad n \geq 0.$$

We use this recurrence relation to find

$$c_2 = -c_0, \;\; c_4 = \frac{c_0}{1\cdot 3}, \;\; c_6 = -\frac{c_0}{1\cdot 3\cdot 5}, \;\; c_8 = \frac{c_0}{1\cdot 3\cdot 5\cdot 7}, \cdots$$

and generally

$$c_{2n} = \frac{(-1)^n c_0}{(1)(3)(5)\cdots(2n-1)}, \quad n \geq 1.$$

Also we find

$$c_3 = -\frac{c_1}{2}, \;\; c_5 = \frac{c_1}{2\cdot 4}, \;\; c_7 = -\frac{c_1}{2\cdot 4\cdot 6}, \cdots$$

and generally

$$c_{2n+1} = \frac{(-1)^n c_1}{(2)(4)(6)\cdots(2n)}, \quad n \geq 1.$$

Since c_0 and c_1 are left arbitrary, setting $c_0 = 0$ and $c_1 = 1$ results in $y = \sum_{n=0}^{\infty} c_n x^n$ becoming

$$y_1(x) = \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2)(4)(6)\cdots(2n)}x^{2n+1}.$$

Letting $c_0 = 1$ and $c_1 = 0$ gives

$$y_2(x) = \sum_{n=0}^{\infty} c_{2n}x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(1)(3)(5)\cdots(2n-1)}x^{2n}.$$

The general solution (not asked for here) is $y = d_1 y_1 + d_2 y_2$ for arbitrary d_1, d_2.