Math 250 Exam #2 Key (Summer 2024)

1 We find $(N_x - M_y)/M = -2/y$, so $\mu(y) = e^{-\int 2/y\,dy} = y^{-2}$ is an integrating factor. Multiplying the DE by this gives $(1 + 2x/y)\,dx - (x^2/y^2)\,dy = 0$, which is exact. There is a function $F(x,y)$ such that $F_x = 1 + 2x/y$ and $F_y = -x^2/y^2$. So

$$F(x,y) = \int \left(1 + \frac{2x}{y}\right) dx = x + \frac{x^2}{y} + g(y).$$

Now from $F_y = -x^2/y^2$ we get $-x^2/y^2 + g'(y) = -x^2/y^2$, so that $g'(y) = 0$, and hence $g(y)$ is constant. We can let $g(y) = 0$. Solution to DE is $F(x,y) = c$, or $x + x^2/y = c$. Also $y \equiv 0$ is a solution.

2 Let $u = y'$, so DE becomes $x^2u' + u^2 = 0$. This is separable, yielding $u = x/(cx-1)$, and hence$y' = x/(cx-1)$. This also is separable:

$$\int dy = \int \frac{x}{cx-1} dx$$

If $c = 0$ we get $y = -\frac{1}{2}x^2 + k$, a one-parameter family of solutions. If $c \neq 0$ we get

$$y = \frac{x}{c} + \frac{\ln|cx-1|}{c^2} + k,$$

a two-parameter family of solutions.

3 Let $u = y'$, so $y'' = u\frac{du}{dy}$. DE becomes $u\frac{du}{dy} + uy = 0$, so either $u \equiv 0$ or $\frac{du}{dy} = -y$, and so either $y \equiv 1$ (using $y(0) = 1$) or $u = -\frac{1}{2}y^2 + c_1$. But $y \equiv 1$ violates $y'(0) = -1$, so the only option is $y' = -\frac{1}{2}y^2 + c_1$, and using $y'(0) = -1$ we find $c_1 = -\frac{1}{2}$. Now, from $y' = -\frac{1}{2}y^2 - \frac{1}{2}$ we separate variables to get

$$x = -2\int \frac{1}{y^2+1}\,dy = -2\tan^{-1} y + c.$$

Again using $y(0) = 1$, we find $c = \frac{\pi}{2}$, and therefore $x = \frac{\pi}{2} - 2\tan^{-1} y$.

4a $y = c_1e^{(-2-\sqrt{5})t} + c_2e^{(-2+\sqrt{5})t}$

4b $y = e^{3t/4}\left[c_1 \cos \frac{\sqrt{23}}{4}t + c_2 \sin \frac{\sqrt{23}}{4}t\right]$

5 Auxiliary equation $r^2 + r + 1 = 0$ gives $r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so

$$y_h(x) = e^{-x/2}(c_1 \cos \tfrac{\sqrt{3}}{2}x + c_2 \sin \tfrac{\sqrt{3}}{2}x)$$

is the general solution to $y'' + y + y = 0$. A particular solution to the given DE has the form $y_p(x) = A\cos x + B\sin x$. Putting this into the DE, we find $A = -1$ and $B = 0$, and so $y_p(x) = -\cos x$. The general solution to the DE is

$$y = y_p + y_h = -\cos x + e^{-x/2}(c_1 \cos \tfrac{\sqrt{3}}{2}x + c_2 \sin \tfrac{\sqrt{3}}{2}x).$$

Finding the solution to the given IVP turns out to be a lot of menial work, despite it having been poached from among the "easy" problems in another textbook, so I'll forgive anyone who does not bother.

6 The general solution to $y'' + 2y' + 5y = 0$ is $y_h = e^{-t}(c_1 \cos 2x + c_2 \sin 2x)$, so linearly independent solutions to the homogeneous DE are $y_1 = e^{-x} \cos 2x$ and $y_2 = e^{-x} \sin 2x$. With $f(x) = e^{-x} \sec 2x$, we find that

$$\int \frac{-f(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} = -\int \frac{\tan 2x}{2}\, dx = -\frac{1}{4} \ln|\sec 2x|,$$

and

$$\int \frac{f(x)y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} = \int \frac{1}{2}\, dx = \frac{x}{2}.$$

A particular solution to the nonhomogeneous DE is thus

$$y_p = e^{-x} \cos 2x \cdot \frac{-1}{4} \ln|\sec 2x| + e^{-x} \sin 2x \cdot \frac{x}{2}$$

The general solution is

$$y = \frac{e^{-x}}{2}\left(x \sin 2x - \frac{\ln|\sec 2x| \cos 2x}{2}\right) + e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$$

7 Standard form for DE is

$$y'' - \frac{2x+1}{x} y' + \frac{x+1}{x} y = 0.$$

So, a 2nd solution to the DE is

$$y_2 = e^x \int \frac{e^{\int \frac{2x+1}{x}\, dx}}{e^{2x}}\, dx = e^x \int |x|\, dx.$$

If we assume $x > 0$, then $y_2(x) = \frac{1}{2}x^2 e^x$.