Math 250 Exam #1 Key (Summer 2024)

1 Let $y = x^m$, so $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Put all this in the DE to get

$$x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m = 0 \quad \hookrightarrow \quad (m^2 - 8m + 15)x^m = 0.$$

This can only be satisfied for all x in an interval I if $m^2 - 8m + 15 = 0$, which is the case only if $m = 3$ or $m = 5$. Thus $y = x^m$ is a solution to the DE only if $m \in \{3, 5\}$.

2 Equation is separable, yielding

$$\int y^{-2}\,dy = \int -2x\,dx \quad \hookrightarrow \quad -\frac{1}{y} = -x^2 + c \quad \hookrightarrow \quad y = \frac{1}{x^2 + c}.$$

3 Separable:

$$\int \frac{1}{y}\,dy = \int \frac{1-x}{x^2}\,dx \quad \hookrightarrow \quad \ln|y| = -\frac{1}{x} - \ln|x| + c.$$

Using $y(-1) = -1$ gives $\ln|-1| = 1 - \ln|-1| + c$, or $c = -1$, so that

$$\ln|y| = -\frac{1}{x} - \ln|x| - 1.$$

Since solution curve contains a point (x, y) where $x, y < 0$, we can let $|x| = -x$ and $|y| = -y$ and write

$$\ln(-y) = -\frac{1}{x} - \ln(-x) - 1 \quad \text{or} \quad y = \frac{1}{xe^{1/x+1}}.$$

4 Get standard form: $y' + \frac{x+2}{x}y = \frac{e^x}{x^2}$. An integrating factor is

$$\mu(x) = \exp\left(\int \frac{x+2}{x}\,dx\right) = e^{x+2\ln|x|} = x^2e^x.$$

Multiply standard form by x^2e^x to get $x^2e^xy' + x(x+2)e^xy = e^{2x}$, or $(x^2e^xy)' = e^{2x}$. So, $x^2e^xy = \frac{1}{2}e^{2x} + c$, and therefore

$$y = \frac{e^{2x} + c}{2x^2e^x}.$$

5 Let $u = y^{-1}$, so $y = 1/u$ and $y' = -u'/u^2$. Then DE becomes $u' + u = -e^x$, which has integrating factor $\mu(x) = e^x$, so write $e^xu' + e^xu = -e^{2x}$, which then becomes $(ue^x)' = -e^{2x}$, and hence $ue^x = -\frac{1}{2}e^{2x} + c$. Thus $y^{-1} = -\frac{1}{2}e^x + ce^{-x}$, or equivalently

$$y = \frac{2}{ce^{-x} - e^x}.$$

6 The equation is given to be exact, with

$$M(x, y) = 2xy^4 + \sin y \quad \text{and} \quad N(x, y) = 4x^2y^3 + x\cos y.$$

We find function F such that $F_x = M$ and $F_y = N$. The former equation gives

$$F(x, y) = \int M(x, y)\,dx = \int (2xy^4 + \sin y)dx = x^2y^4 + x\sin y + g(y).$$

Now we have

$$4x^2y^3 + x\cos y = N(x,y) = F_y(x,y) = 4x^2y^3 + x\cos y + g'(y) \quad \hookrightarrow \quad g'(y) = 0,$$

so that $g(y) = \hat{c}$ for arbitrary constant $\hat{c}$. This leaves us with

$$F(x,y) = x^2y^4 + x\sin y + \hat{c}.$$

The general (implicit) solution to the DE is given by $F(x,y) = \tilde{c}$ for arbitrary constant $\tilde{c}$, which here becomes $x^2y^4 + x\sin y + \hat{c} = \tilde{c}$. Letting $c = \tilde{c} - \hat{c}$, we finally get

$$x^2y^4 + x\sin y = c.$$

7a We have $(x^2 - 2y^2)' = 0$, giving $2x - 4yy' = 0$, and finally $y' = x/2y$.

7b Solve $y' = -2y/x$, which is separable and yields $\ln|y| = -2\ln|x| + \hat{c}$. Thus $|y| = e^{\hat{c}}/x^2 = C/x^2$ for $C > 0$ arbitrary. Then $y = \pm C/x^2 = c/x^2$ for $c \neq 0$ arbitrary. But $y \equiv 0$ is also a solution, so the orthogonal trajectories are $y = c/x^2$ for $c \in (-\infty, \infty)$.

8 Rewrite the DE as

$$\frac{dy}{dx} = \frac{(y/x)^3 - 1}{(y/x)^2}.$$

Let $y = xu$, so that $u = y/x$ and DE becomes

$$\frac{d}{dx}(xu) = \frac{u^3 - 1}{u^2} \quad \hookrightarrow \quad x\frac{du}{dx} + u = u - \frac{1}{u^2} \quad \hookrightarrow \quad -\int u^2\,du = \int \frac{1}{x}\,dx.$$

This solves to give $-\frac{1}{3}u^3 = \ln|x| + c$, and hence $-\frac{y^3}{3x^3} = \ln|x| + c$. Using $y(1) = 2$ results in $c = -\frac{8}{3}$, and so

$$-\frac{y^3}{3x^3} = \ln x - \frac{8}{3} \quad \text{or} \quad y^3 = 8x^3 - 3x^3\ln x \quad \text{or} \quad y = x\sqrt[3]{8 - 3\ln x}.$$