MATH 250 EXAM #4 KEY (SUMMER 2022)

1 In operator notation the two equations are $(2D-5)x + Dy = e^t$ and $(D-1)x + Dy = 5e^t$. From the first equation comes $Dy = (5-2D)x + e^t$, which when substituted into the second equation yields $(4-D)x = 4e^t$. We solve this for x to get $x = c_1e^{4t} + \frac{4}{3}e^t$, which when put into $Dy = (5-2D)x + e^t$ yields $y = -\frac{3}{4}c_1e^{4t} + c_2 + 5e^t$. The system's general solution is thus

$$x(t) = c_1e^{4t} + \frac{4}{3}e^t, \quad y(t) = -\frac{3}{4}c_1e^{4t} + c_2 + 5e^t.$$

2 The Laplace transform of the system yields the equations $sX = -X + Y$ and $sY - 1 = 2X$. Solving for $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$, we have

$$\mathcal{L}[x] = \frac{1}{(s-1)(s+2)}, \quad \mathcal{L}[y] = \frac{1}{s} + \frac{2}{s(s-1)(s+2)},$$

and therefore

$$x(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}, \quad y(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}.$$

3 The system is

$$\begin{cases} x' = \frac{1}{50}y - \frac{3}{50}x \\ y' = \frac{3}{50}x - \frac{7}{100}y + \frac{1}{100}z \\ z' = \frac{1}{20}y - \frac{1}{20}z \end{cases}$$

4 Generally the system has the form

$$\begin{cases} m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) \\ m_2x_2'' = -k_2(x_2 - x_1) - k_3x_2 \end{cases}$$

5 The general solution has the form

$$y = c_0\left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 - \frac{9}{320}x^5 + \cdots\right) + c_1\left(x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots\right),$$

but with the initial conditions $y(0) = 0$ and $y'(0) = 1$ we find that $c_0 = 0$ and $c_1 = 1$, and so the IVP's solution is

$$y = x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots.$$