Math 250 Exam #3 Key (Summer 2022)

1 Letting $Y = \mathcal{L}[y]$, we have

$$sY - y(0) - 2Y = \frac{1}{s-2} \quad \hookrightarrow \quad sY - c - 2Y = \frac{1}{s-2} \quad \hookrightarrow \quad Y(s) = \frac{1}{(s-2)^2} + \frac{c}{s-2}.$$

Thus

$$y(x) = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right] + c\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = xe^{2x} + ce^{2x} = (x+c)e^{2x}.$$

2 Letting $Y = \mathcal{L}[y]$, we transform the ODE, obtaining $\mathcal{L}[y''] + 4\mathcal{L}[y'] - 5\mathcal{L}[y] = \mathcal{L}[xe^x]$, and hence

$$(s^2Y - s) + 4(sY - 1) - 5Y = \frac{1}{(s-1)^2}.$$

Solve for Y and apply partial fraction decomposition:

$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{12}\left(\frac{1}{(s-1)^2}\right)$$

Apply the inverse Laplace transform, obtaining

$$y(x) = \frac{35}{216}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] + \frac{181}{216}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{36}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{12}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right],$$

and hence

$$y(x) = \frac{35}{216}e^{-5x} + \frac{181}{216}e^x - \frac{1}{36}xe^x + \frac{1}{12}x^2e^x.$$

3a The time scale may be chosen so that $x(0) = 0$, and thus $x(0.4n) = 0$ for every integer $n \geq 0$. Also we have $|x'(0.4n)| = 6$ for $n \geq 0$, and we can choose the space scale so that $x'(0) = 6$, $x'(0.4) = -6$, and so on. By definition of simple harmonic motion,

$$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t,$$

but $x(0) = 0$ immediately implies $c_1 = 0$, so that $x(t) = c_2 \sin\omega_0 t$. The period of motion is $T = 2(0.4) = 0.8$, but also $T = 2\pi/\omega_0$ so that we must have $\omega_0 = 5\pi/2$. Now we have $x(t) = c_2\sin(\frac{5\pi}{2}t)$, and thus

$$x'(t) = \frac{5\pi}{2}c_2\cos\left(\frac{5\pi}{2}t\right).$$

Using $x'(0) = 6$ we find $c_2 = \frac{12}{5\pi}$, and therefore

$$x(t) = \frac{12}{5\pi}\sin\left(\frac{5\pi}{2}t\right).$$

3b Period is $T = \frac{4}{5}$ sec; natural frequency is $\nu = 1/T = \frac{5}{4}$ cycles/sec; amplitude is $A = \frac{12}{5\pi}$ m.

4a We're given initial conditions $x(0) = \frac{1}{2}$ and $x'(0) = \sqrt{2}$, and also $m = \frac{1}{8}$, $k = 16$, and $b = 0$. The ODE has the form $\frac{1}{8}x'' + 16x = 0$, which has general solution

$$x(t) = c_1\cos(8\sqrt{2}t) + c_2\sin(8\sqrt{2}t).$$

Using the initial conditions, we obtain the equation of motion

$$x(t) = \frac{1}{2}\cos(8\sqrt{2}t) + \frac{1}{8}\sin(8\sqrt{2}t) = \frac{\sqrt{17}}{8}\sin\left(8\sqrt{2}t + \varphi\right),$$

where $\varphi = \arctan(4) \approx 1.326$.

4b Period is $T = \frac{\pi\sqrt{2}}{8}$ sec; natural frequency is $\nu = \frac{8}{\pi\sqrt{2}}$ cycles/sec; amplitude is $A = \frac{\sqrt{17}}{8}$ m.

4c The times when $x(t) = 0$ are times $t > 0$ when $8\sqrt{2}t + \varphi = n\pi$ for some integer n. That is,

$$t = \frac{n\pi - \varphi}{8\sqrt{2}} \approx \frac{n\pi - 1.326}{8\sqrt{2}}.$$

We find the smallest value of n such that $t > 0$, which is $n = 1$. Then we get

$$t = \frac{\pi - \varphi}{8\sqrt{2}} \approx 0.16 \text{ sec.}$$

4d Maximum displacement occurs at times $t > 0$ when $x'(t) = \sqrt{34}\cos(8\sqrt{2}t + \varphi) = 0$, which are times $t > 0$ when $8\sqrt{2}t + \varphi = \frac{\pi}{2} + n\pi$. That is,

$$t = \frac{1}{8\sqrt{2}}\left(\frac{(2n+1)\pi}{2} - \varphi\right). \tag{1}$$

There is no damping, so the maximum displacement is reached twice each period, with the first time of maximum displacment being given by (1) when $n = 0$. Then we get

$$t = \frac{1}{8\sqrt{2}}\left(\frac{\pi}{2} - \varphi\right) \approx 0.02164.$$

The maximum displacement is $x(0.02164) = 0.515$ m.