Math 250 Exam #2 Key (Summer 2022)

1 Let $x(t)$ be the number of mice at time t when cats are not a factor and only natural increase is considered. The growth model is $x' = kx$, and if $x(0) = n$ then the given doubling rate implies that $x(1) = 2n$. Solving yields

$$\frac{dx}{dt} = kx \quad \hookrightarrow \quad \int \frac{1}{x}\,dx = \int k\,dt \quad \hookrightarrow \quad \ln x = kt + c \quad \hookrightarrow \quad x(t) = Ce^{kt},$$

and with $x(0) = n$ we find $C = n$, so that $x(t) = ne^{kt}$. With $x(1) = 2n$ we then get

$$2n = x(1) = ne^k \quad \hookrightarrow \quad e^k = 2 \quad \hookrightarrow \quad k = \ln 2,$$

and so $x' = x \ln 2$. This is the rate of change in the population at time t due solely to natural increase.

Now let $y(t)$ be the number of mice present when cats are a factor, so $t = 0$ marks the beginning of 1990 and $y(0) = 100,000$. The rate of change of the population, y', is $y \ln 2$ due to natural increase plus -1000 due to the cats. The model is

$$\frac{dy}{dt} = y \ln 2 - 1000.$$

This ODE tells us that the rate of change at time t is $\frac{1}{4}y(t)$ bacteria per day minus n bacteria per day. Solving this separable equation, we obtain

$$\int \frac{1}{y \ln 2 - 1000}\,dy = \int dt \quad \hookrightarrow \quad \frac{\ln|y \ln 2 - 1000|}{\ln 2} = t + c \quad \hookrightarrow \quad y = \frac{1000 + C_0 2^t}{\ln 2},$$

where $y(0) = 100,000$ leads to $C_0 = 100,000 \ln 2 - 1000$, so that the number of mice at time t is

$$y(t) = \frac{1000[1 + (100 \ln 2 - 1)2^t]}{\ln 2}.$$

At time $t = 24$ the number of mice is $y(24) = 1.65 \times 10^{12}$. The cats aren't putting much of a brake on the mouse population!

2 Characteristic equation is $r^2 - 2r - 3 = 0$, so $r = -1, 3$, and the general solution is $y = c_1 e^{-x} + c_2 e^{3x}$. Using the initial conditions we find that $c_1 = -\frac{5}{4}$ and $c_2 = \frac{1}{4}$. The solution to the IVP is

$$y(x) = -\frac{5}{4}e^{-x} + \frac{1}{4}e^{3x}.$$

3 The characteristic equation $r^4 + 8r^2 + 16 = 0$ has roots $r = \pm 2i$ with multiplicity 2. Complex-valued solutions to the ODE are thus e^{2ix} and xe^{2ix}; but since $e^{i\theta} = \cos\theta + i \sin\theta$, and the real and imaginary parts of a complex solution are themselves solutions, we find $\cos 2x$, $\sin 2x$, $x \cos 2x$, and $x \sin 2x$ to be linearly independent real solutions. General solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x.$$

4 The characteristic equation $r^5 - 2r^4 + r^3 = 0$ factors as $r^3(r-1)^2 = 0$, so $r = 0$ is a root (multiplicity 3) and $r = 1$ is a root (multiplicity 2). General solution:

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 x e^x.$$

5 The characteristic equation $r^2+2r+10=0$ has roots $-1\pm 3i$, so complementary function is $y_c=(c_1\cos 3x+c_2\sin 3x)e^{-x}$. A particular solution to the ODE has the form $y_p=(Ax+B)e^{-2x}$. Putting y_p and its derivatives into the ODE and applying the undetermined coefficients method yields $A=\frac{1}{2}$ and $B=\frac{1}{10}$. The general solution is therefore

$$y(x)=(c_1\cos 3x+c_2\sin 3x)e^{-x}+\left(\tfrac{1}{2}x+\tfrac{1}{10}\right)e^{-2x}.$$

6 The characteristic equation $r^3+r=0$ has roots $r=0,\pm i$, so complementary function is $y_c=c_1+c_2\cos x+c_3\sin x$. A particular solution to $y'''+y'=2x^2$ has the form $y_{p_1}=Ax^3+Bx^2+Cx$, and putting y_{p_1} and its derivatives into $y'''+y'=2x^2$ yields $y_{p_1}=\frac{2}{3}x^3-4x$. A particular solution to $y'''+y'=4\sin x$ has the form $y_{p_2}=x(A\cos x+B\sin x)$, and putting y_{p_2} and its derivatives into $y'''+y'=4\sin x$ yields $A=0$ and $B=-2$, so that $y_{p_2}=-2x\sin x$. By the superposition principle the general solution to the original ODE is

$$y(x)=c_1+c_2\cos x+c_3\sin x+\tfrac{2}{3}x^3-4x-2x\sin x.$$

7 Since the characteristic equation $r^2+6r+9=0$ has double root -3, two linearly independent solutions to the corresponding homogeneous equation are $y_1=e^{-3x}$ and $y_2=xe^{-3x}$. The Wronskian of y_1 and y_2 is thus

$$\mathcal{W}[y_1,y_2](x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & (1-3x)e^{-3x} \end{vmatrix}=e^{-6x}.$$

Now, by variation of parameters,

$$u_1(x)=-\int\frac{1}{x^2}\,dx=\frac{1}{x}+C_1 \quad\text{and}\quad u_2(x)=\int\frac{1}{x^3}\,dx=-\frac{1}{2x^2}+C_2.$$

Since u_1 and u_2 are used to construct a *particular* solution $y_p=u_1y_1+u_2y_2$ to the ODE, we may choose to let $C_1=C_2=0$. The general solution is

$$y(x)=c_1e^{-3x}+c_2xe^{-3x}+\frac{e^{-3x}}{2x}=\left(c_1+c_2x+\frac{1}{2x}\right)e^{-3x}.$$