Math 250 Exam #1 Key (Summer 2022)

1 Simplify the ODE a bit, writing $x^2+y^2+2cy=1$. Differentiate this with respect to x to get $2x+2yy'+2cy'=0$, and thus $c=-x/y'-y$. This the expression for c into the simplified ODE to get

$$x^2+y^2+2\left(-\frac{x}{y'}-y\right)y=1 \quad\hookrightarrow\quad (x^2-y^2)y'=2xy+y'.$$

2 Equation is separable, yielding

$$\int(3y^2+e^y)\,dy=\int\cos x\,dx \quad\hookrightarrow\quad y^3+e^y=\sin x+c.$$

With the initial condition $y(0)=2$ we find $c=8+e^2$, and so $y^3+e^y=\sin x+8+e^2$.

3 Divide by $2x^2$ to write the ODE as

$$\frac{dy}{dx}=\frac{1}{2}\left[1+\left(\frac{y}{x}\right)^2\right].$$

Letting $y=ux$, the ODE becomes separable:

$$x\frac{du}{dx}+u=\frac{1}{2}(1+u^2) \quad\hookrightarrow\quad \int\frac{1}{(u-1)^2}\,du=\int\frac{1}{2x}\,dx \quad\hookrightarrow\quad -\frac{1}{u-1}=\frac{1}{2}\ln|x|+c.$$

Thus

$$u=1-\frac{2}{\ln|x|+c} \quad\hookrightarrow\quad \frac{y}{x}=1-\frac{2}{\ln|x|+c} \quad\hookrightarrow\quad y=x-\frac{2x}{\ln|x|+c},$$

with the solution having intervals of validity $(-\infty,-e^{-c})$, $(-e^{-c},0)$, $(0,e^{-c})$, and (e^{-c},∞).

4 The equation is given to be exact, with

$$M(x,y)=2x+e^y \quad\text{and}\quad N(x,y)=xe^y.$$

We find function F such that $F_x=M$ and $F_y=N$. The former equation gives

$$F(x,y)=\int F_x(x,y)\,dx=\int M(x,y)\,dx=\int(2x+e^y)dx=x^2+xe^y+g(y).$$

Differentiating this result respect to y then yields

$$F_y(x,y)=xe^y+g'(y),$$

with $F_y=N$ implying that

$$xe^y+g'(y)=xe^y.$$

Thus $g'(y)=0$, and so $g(y)=c_1$ for some arbitrary constant c_1. This leaves us with

$$F(x,y)=x^2+xe^y+c_1.$$

The general (implicit) solution to the ODE is given by $F(x,y)=c_2$ for arbitrary constant c_2, which here becomes

$$x^2+xe^y+c_1=c_2.$$

Letting $c=c_2-c_1$, we finally write $x^2+xe^y=c$.

5 Standard form is $y' - \frac{3}{x}y = x^2$. A suitable integrating factor is

$$\mu(x) = e^{-\int \frac{3}{x}dx} = e^{-3\ln|x|+c} = |x|^{-3},$$

where we choose $c = 0$. Since the solution to the IVP must contain the point $(1, 0)$, so that $x > 0$, we have $\mu(x) = x^{-3}$. Multiply standard form by this to get

$$\frac{1}{x^3}y' - \frac{3}{x^4}y = \frac{1}{x} \quad\hookrightarrow\quad \left(\frac{y}{x^3}\right)' = \frac{1}{x} \quad\hookrightarrow\quad \frac{y}{x^3} = \int \frac{1}{x}\,dx \quad\hookrightarrow\quad y = x^3 \ln x + cx^3.$$

Using $y(1) = 0$ we obtain $c = 0$, and therefore $y = x^3 \ln x$.

6 Let $u = y^{-1}$ to transform the ODE into the linear equation

$$\frac{du}{dx} - \frac{3}{x}u = -x^2. \tag{1}$$

The integrating factor is $\mu(x) = |x|^{-3}$. If $x > 0$, so that $\mu(x) = x^{-3}$, then multiplying (1) by $\mu(x)$ yields

$$x^{-3}u' - 3x^{-4}u = -\frac{1}{x} \quad\hookrightarrow\quad (x^{-3}u)' = -\frac{1}{x} \quad\hookrightarrow\quad x^{-3}u = -\ln x + c,$$

and so $u = x^3(c - \ln x)$. If $x < 0$, so that $\mu(x) = -x^{-3}$, then multiplying (1) by $\mu(x)$ yields much the same result, only $u = x^3(c - \ln(-x))$. We combine the two families of solutions by writing $u = x^3(c - \ln|x|)$. Finally, since $y = 1/u$ we obtain

$$y = \frac{1}{cx^3 - x^3 \ln|x|}.$$

7 Let $x(t)$ be the amount of isopropyl alcohol in kilograms in the tank at time t, so that $x(0) = 40$, and dx/dt is the rate of change of the amount. Noting that the volume of solution in the tank at time t is $500 + t$, we have

$$\frac{dx}{dt} = (\text{rate going in}) - (\text{rate going out}) = 0 - \left(\frac{2\text{ L}}{1\text{ min}}\right)\left(\frac{x}{500+t}\right),$$

or simply

$$\frac{dx}{dt} = -\frac{2x}{t+500}.$$

The equation is separable:

$$-\int \frac{1}{2x}\,dx = \int \frac{1}{t+500}\,dt \quad\hookrightarrow\quad -\frac{1}{2}\ln x = \ln(t+500) + c \quad\hookrightarrow\quad x(t) = \frac{C}{(t+500)^2},$$

where $C := e^c > 0$ is arbitrary. Using $x(0) = 40$ we find that $40 = C/500^2$, or $C = 10^7$, and so

$$x(t) = \frac{10^7}{(t+500)^2}.$$

Now we find t when $x(t) = 25$:

$$25 = \frac{10^7}{(t+500)^2} \quad\hookrightarrow\quad (t+500)^2 = \frac{10^7}{25} \quad\hookrightarrow\quad t = 200\sqrt{10} - 500 \approx 132.5\text{ min}.$$

8 Let $T(t)$ be the temperature of the object at time t. We're given $T(0) = 10$ and $T(10) = 30$, and the ambient temperature is $M = 80$. With Newton's Law of Warming we have

$$\frac{dT}{dt} = k(T - 80) \quad \hookrightarrow \quad \int \frac{1}{T-80}\,dT = \int k\,dt \quad \hookrightarrow \quad \ln(80 - T) = kt + c,$$

where $\ln|T - 80| = \ln(80 - T)$ since $T(t) < 80$ for all $t \geq 0$ according to the model. Solving for T yields

$$T(t) = 80 - Ce^{kt},$$

and with $T(0) = 10$ we discover that $C = 70$, so

$$T(t) = 80 - 70e^{kt}.$$

Using $T(10) = 30$ allows us to solve for k to get $k = \frac{1}{10}\ln\frac{5}{7} \approx -0.03365$. Therefore

$$T(t) = 80 - 70e^{-0.03365t}.$$

Finally, the temperature of the object after 30 minutes is

$$T(30) = 80 - 70e^{-0.03365(30)} \approx 54.5°\text{F}.$$

9 Suppose that $g \equiv kf$ for constant k on I. Suppose further that $c_1 f + c_2 g \equiv 0$ on I. Then

$$c_1 f + c_2 g = c_1 f + c_2 k f = (c_1 + c_2 k) f \equiv 0$$

on I. If $k = 0$ we can satisfy the identity with $c_1 = 0$ and $c_2 = 1$. If $k \neq 0$ then let $c_1 = 1$ and $c_2 = -1/k$ to safisfy the identity. In either case we can satisfy the identity on I without having $c_1 = c_2 = 0$. Therefore f and g are linearly dependent on I.