Math 250 Exam #2 Key (Summer 2017)

1 Standard form of equation:

$$y'' + \frac{2}{t}y' - \frac{6}{t^2} = 0.$$

Reduction of order formula gives

$$y_2(t) = t^2 \int \frac{e^{-\int 2/t\,dt}}{(t^2)^2}\,dt = t^2 \int \frac{1}{t^6}\,dt = -\frac{1}{5t^3}.$$

Any constant multiple of this solution is also a solution, so we can also have $y_2(t) = t^{-3}$.

2a Auxiliary equation: $r^2 + 8r + 16 = 0$. This has -4 as a double root, and so

$$y(x) = c_1 e^{-4x} + c_2 x e^{-4x}$$

is the general solution to the ODE.

2b Auxiliary equation: $3r^3 + 10r^2 + 15r + 4 = 0$. With synthetic division and the remainder theorem from algebra we find that $-1/3$ is a root, and moreover

$$(3r+1)(r^2+3r+4) = 0.$$

is a factorization of the polynomial. From $r^2 + 3r + 4 = 0$ we obtain the complex-valued roots $-\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$. General solution to the ODE:

$$y = c_1 e^{-x/3} + e^{-3x/2}\big(c_2 \cos \tfrac{\sqrt{7}}{2}x + c_2 \sin \tfrac{\sqrt{7}}{2}x\big).$$

3 Auxiliary equation: $r^2 - 6r + 25 = 0$, which has solutions

$$r = \frac{6 \pm \sqrt{6^2 - 4(25)}}{2} = 3 \pm 4i.$$

General solution to ODE:

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x),$$

and hence

$$y'(x) = 3e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x).$$

with the initial conditions given we find that $c_1 = 3$ and $c_2 = -2$. Therefore the solution to the IVP is

$$y(x) = e^{3x}(3\cos 4x - 2\sin 4x).$$

4 Roots to the auxiliary equation must be 0 and $2 \pm 5i$. Auxiliary equation:

$$r\big[r - (2+5i)\big]\big[r - (2-5i)\big] = 0 \quad \Rightarrow \quad r^3 - 4r^2 + 29r = 0.$$

The ODE with this auxiliary equation:

$$y''' - 4y'' + 29y' = 0.$$

5 Standard form is $y'' - 7x^{-1}y' + 16x^{-2}y = 0$. A second particular solution:

$$y_2(x) = x^4 \int \frac{e^{-\int(-7/x)\,dx}}{(x^4)^2} dx = x^4 \int \frac{e^{7\ln x}}{x^8}\, dx = x^4 \int \frac{1}{x}\, dx = x^4 \ln x.$$

6a Auxiliary equation is $r^3 - 3r^2 + 3r - 1 = 0$ has root 1 with multiplicity 3, and since the nonhomogeneity has form $P_m(x)e^{\alpha x}$ with $\alpha = 1$ (and $m = 0$), we have $s = 3$ in the form for the particular solution given by the Method of Undetermined Coefficients:

$$y_p(x) = x^s e^{\alpha x} \sum_{k=0}^{m} A_k x^k = Ax^3 e^x \tag{1}$$

Now,

$$y_p'(x) = (3Ax^2 + Ax^3)e^x,$$
$$y_p''(x) = (6Ax + 6Ax^2 + Ax^3)e^x,$$
$$y_p'''(x) = (6A + 18Ax + 9Ax^2 + Ax^3)e^x,$$

which when put into the ODE results in $6A = -4$, and thus $A = -\frac{2}{3}$. Therefore $y_p(x) = -\frac{2}{3}x^3e^x$.

6b Given that 1 is a root of the auxiliary equation with multiplicity 3,

$$y(x) = -\frac{2}{3}x^3e^x + c_1e^x + c_2xe^x + c_3x^2e^x$$

is the general solution.

7a Nonhomogeneity $f(x) = x^2 - 2x$ has form $P_m(x)e^{\alpha x}$ with $m = 2$ and $\alpha = 0$. The auxiliary equation is

$$\frac{1}{4}r^2 + r + 1 = 0,$$

which has root -2 with multiplicity 2. Since $\alpha = 0$ is not a root, we have $s = 0$ in the form (1) for the particular solution, so that $y_p(x) = A_1 + A_2x + A_3x^2$. Rewrite this as $y_p(x) = A + Bx + Cx^2$. Then

$$y_p'(x) = B + 2Cx \quad \text{and} \quad y_p''(x) = 2C.$$

Putting these results into the ODE (multiplied by 4) gives

$$4Cx^2 + (4B + 8C)x + (4A + 4B + 2C) = 4x^2 - 8x,$$

which leads to the system

$$\begin{cases} 4C = 4 \\ 4B + 8C = -8 \\ 4A + 4B + 2C = 0 \end{cases}$$

The solution is $A = \frac{7}{2}$, $B = -4$, $C = 1$. Particular solution is therefore

$$y_p(x) = x^2 - 4x + \frac{7}{2}$$

7b With -2 a double root of the auxiliary equation, the general solution is

$$y(x) = x^2 - 4x + \frac{7}{2} + c_1 e^{-2x} + c_2 x e^{-2x}.$$

8 Auxiliary equation $r^2 - 2r + 2 = 0$ has roots $1 \pm i$, and so $y_1(x) = e^x \cos x$ and $y_2(x) = e^x \sin x$ form a fundamental solution set for $y'' - 2y' + 2y = 0$. We then find that $\mathcal{W}[y_1, y_2](x) = e^{2x}$. Now,

$$\begin{aligned} u_1(x) &= \int \frac{-e^x \sin x \cdot e^x \tan x}{e^{2x}}\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx \\ &= \int (\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|, \end{aligned}$$

and

$$u_2(x) = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\,dx = \int \sin x\,dx = -\cos x.$$

Therefore

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) = \big(\sin x - \ln|\sec x + \tan x|\big)e^x \cos x - e^x \sin x \cos x \\ &= -e^x \cos x \ln|\sec x + \tan x|. \end{aligned}$$

General solution:

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x - e^x \cos x \ln|\sec x + \tan x|.$$

9 Let $u = y'$, so $u' = y''$ and equation becomes $uu' = 4x$. This is separable, giving

$$\int u\,du = \int 4x\,dx \quad\Rightarrow\quad u^2 = 4x^2 + c \quad\Rightarrow\quad (y')^2 = 4x^2 + c \quad\Rightarrow\quad |y'| = \sqrt{4x^2 + c}.$$

Since $y'(1) = 2 > 0$, we have $c = 0$ and $|y'| = y'$, so that $y' = \sqrt{4x^2} = 2x$. (We also can assume $x > 0$ since $x > 0$ in the initial conditions.) Thus $y = x^2 + d$, and with $y(1) = 5$ we obtain $d = 4$, and finally arrive at the solution $y = x^2 + 4$.

10 We have $y'' = 1 - y^2$, so

$$y''' = -2yy', \quad y^{(4)} = -2yy'' - 2(y')^2, \quad y^{(5)} = -6y'y'' - 2yy'''.$$

We then use $y(0) = 2$ and $y'(0) = 3$ to obtain $y''(0) = -3$, $y'''(0) = -12$, $y^{(4)}(0) = -6$, and $y^{(5)}(0) = 102$. Taylor series approximations of the solution to the ODE:

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!} + \frac{y^{(5)}(0)}{5!} \\ &= 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5. \end{aligned}$$

11a IVP is $y'' + 10y' + 16y = 0$, $y(0) = 1$, $y'(0) = -12$. Equation of motion:

$$y(t) = -\frac{2}{3}e^{-2t} + \frac{5}{3}e^{-8t}.$$

11b Find smallest $t > 0$ such that $y(t) = 0$. This yields $\frac{5}{3}e^{-8t} = \frac{2}{3}e^{-2t}$, which becomes $e^{6t} = \frac{5}{2}$, and hence $t = \frac{1}{6}\ln\frac{5}{2}$. This is a positive value, approximately 0.153 second.