Math 250 Exam #1 Key (Summer 2017)

1 Substituting x^m for y in the equation gives

$$x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m = 0 \quad \Rightarrow \quad [m(m-1) - 7m + 15]x^m = 0,$$

so that $(m^2 - 8m + 15)x^m = 0$ for all x in some open interval, and thus $m^2 - 8m + 15 = 0$ must hold. This implies $m = 3, 5$.

2 We have

$$y' = \frac{y+x}{y-x}, \quad y(x_0) = y_0,$$

which is an IVP that the Existence-Uniqueness Theorem implies will have a unique solution if

$$f(x,y) = \frac{y+x}{y-x} \quad \text{and} \quad f_y(x,y) = -\frac{2x}{(y-x)^2}$$

are both continuous on an open rectangle containing (x_0, y_0). This is so for any $(x_0, y_0) \in \mathbb{R}^2$ such that $y_0 - x_0 \neq 0$, or equivalently $y_0 \neq x_0$. That is, the IVP will have a unique solution so long as (x_0, y_0) does not lie on the line $y = x$.

3 Amount left to be memorized at time t is $M - A(t)$, and so $A'(t)$ is proportional to $M - A(t)$; that is,

$$\frac{dA}{dt} = k(M - A).$$

4 Separation of variables gives

$$\int 2y\,dy = -\int \frac{\sin 3x}{\cos^3 3x}\,dx = \frac{1}{3}\int \frac{1}{u^3}\,du,$$

where $u = \cos 3x$. Hence

$$y^2 = -\frac{1}{6\cos^2 3x} + c.$$

5a Separation of variables gives

$$\int 4y\,dy = \int (3x-1)\,dx \quad \Rightarrow \quad 2y^2 = \frac{3}{2}x^2 - x + c.$$

With $y(-2) = -1$ we obtain $c = -6$, and so we have

$$y^2 = \tfrac{3}{4}x^2 - \tfrac{1}{2}x - 3 \quad \text{or} \quad |y| = \sqrt{\tfrac{3}{4}x^2 - \tfrac{1}{2}x - 3}.$$

This is good enough for us. However, since $y < 0$ at the initial point $(-2, -1)$, we can resolve the absolute value:

$$y = -\sqrt{\tfrac{3}{4}x^2 - \tfrac{1}{2}x - 3}.$$

5b We must have $\frac{3}{4}x^2 - \frac{1}{2}x - 3 > 0$, or equivalently $x \in \left(-\infty, \frac{1-\sqrt{37}}{3}\right) \cup \left(\frac{1+\sqrt{37}}{3}, \infty\right)$. But since $x < 0$ at the initial point, it follows that the interval of validity is $\left(-\infty, \frac{1-\sqrt{37}}{3}\right)$.

6 We have $y' + (2x-1)y = 4x - 2$. Integrating factor is

$$\mu(x) = e^{\int (2x-1)dt} = e^{x^2-x},$$

which we multiply the ODE by to get

$$e^{x^2-x}\frac{dy}{dx} + (2x-1)e^{x^2-x}y = (4x-2)e^{x^2-x},$$

or

$$\left(e^{x^2-x}y\right)' = (4x-2)e^{x^2-x}.$$

Integrate both sides:

$$e^{x^2-x}y = \int (4x-2)e^{x^2-x}dx = 2e^{x^2-x} + c.$$

Therefore

$$y(x) = 2 + ce^{x-x^2}.$$

7 The equation is separable:

$$\int \frac{L}{E-Ri}\,di = \int dt \quad \Rightarrow \quad -\frac{L}{R}\ln|E-Ri| = t + c \quad \Rightarrow \quad |E-Ri| = e^{-\frac{R}{L}(t+c)} = Ce^{-Rt/L},$$

for $C > 0$, and hence

$$i(t) = \frac{E - Ce^{-Rt/L}}{R}.$$

With the initial condition $i(0) = i_0$ we find that $C = E - i_0R$, and therefore

$$i(t) = \frac{E - (E - i_0R)e^{-Rt/L}}{R}.$$

8 We find a function F such that $F_x(x,y) = e^x + y$ and $F_y(x,y) = 2 + x + ye^y$. Now,

$$F(x,y) = \int (e^x + y)dx = e^x + xy + g(y)$$

for arbitrary differentiable function g. Then

$$2 + x + ye^y = F_y(x,y) = x + g'(y) \quad \Rightarrow \quad g'(y) = 2 + ye^y \quad \Rightarrow \quad g(y) = 2y + ye^y - e^y,$$

so

$$F(x,y) = e^x + xy + 2y + ye^y - e^y.$$

Solution to ODE is $F(x,y) = c$; that is,

$$e^x + xy + 2y + ye^y - e^y = c.$$

Initial condition gives $y = 1$ when $x = 0$, so $1 + 0 + 2 + e - e = c$, or $c = 3$, and therefore the solution to the IVP is

$$e^x + (y-1)e^y + (x+2)y = 3.$$

9 With $M_y = x + 2y + 1$ and $N_x = 1$, we find that $(M_y - N_x)/N = 1$, and so $\mu(x) = e^x$. Multiplying the ODE by this gives an exact equation for which

$$M(x, y) = ye^x(x + y + 1), \quad N(x, y) = e^x(x + 2y).$$

Thus there exists F such that $F_x = M$ and $F_y = N$. With $F_x = M$ we obtain

$$F(x, y) = xye^x + y^2e^x + g(y)$$

for an arbitrary differentiable function $g(y)$. With $F_y = N$ we obtain

$$xe^x + 2ye^x + g'(y) = xe^x + 2ye^x,$$

so that $g'(y) = 0$, and we may let $g(y) = 0$. General implicit solution: $F(x, y) = c$, or

$$xye^x + y^2e^x = c.$$

10 Rewrite the equation as

$$y' = \frac{1 + (y/x)e^{y/x}}{e^{y/x}}.$$

Let $u = y/x$, so $y' = xu' + u$. Equation becomes

$$xu' + u = \frac{1 + ue^u}{e^u} \quad \Rightarrow \quad u' = \frac{1}{xe^u} \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \int e^u\,du \quad \Rightarrow \quad \ln|x| = e^u + c$$

$$\Rightarrow \quad |x| = Ce^{e^u} = Ce^{e^{y/x}}, \quad C > 0.$$

Therefore $x = Ce^{e^{y/x}}$ for $C \neq 0$.

11a Basic model for radioactive decay: $x(t) = x_0e^{-kt}$. Given: $x(0) = 100$. Thus $x_0 = 100$ and we have $x(t) = 100e^{-kt}$. Find k. Given: $x(6) = 96.6$. So $96.6 = 100e^{-kt}$, which solves to give $k \approx 0.00577$. Therefore

$$x(t) = 100e^{-0.00577t}.$$

After 24 hours the quantity of isotope remaining is

$$x(24) = 100e^{-0.00577(24)} \approx 87.1 \text{ mg}.$$

11b Find t for which $x(t) = 10$:

$$100e^{-0.00577t} = 2 \quad \Rightarrow \quad e^{-0.00577t} = 0.02 \quad \Rightarrow \quad t = -\frac{\ln 0.1}{0.00577} \approx 678.0 \text{ hr}.$$

12 Newton's Law of Cooling states that $T'(t) = k[T(t) - M]$, where $M = 68$ is the temperature of the house. From this ODE we obtain

$$\int \frac{1}{T - 68}dT = \int k\,dt \quad \Rightarrow \quad T(t) = 68 + Ce^{kt}.$$

Letting $t = 0$ be the time of death and $t = \tau$ the time of discovery, we're given $T(0) = 98.6$, $T(\tau) = 83$, $T(\tau + 1) = 77$. With $T(0) = 98.6$ we find that $C = 30.6$. So

$$T(t) = 68 + 30.6e^{kt}.$$

Now, with $T(\tau) = 83$ and $T(\tau + 1) = 77$ we obtain

$$83 = 68 + 30.6e^{k\tau} \quad \Rightarrow \quad e^{k\tau} = \frac{25}{51} \tag{1}$$

and

$$77 = 68 + 30.6e^{k(\tau+1)} = 68 + 30.6e^{k\tau}e^{k}, \tag{2}$$

respectively. Substituting (1) into (2) gives

$$77 = 68 + 30.6\left(\frac{25}{51}e^{k}\right),$$

which solves to give $k = \ln 0.6$. Putting this into (1) yields

$$e^{\tau \ln 0.6} = \frac{25}{51} \quad \Rightarrow \quad \tau \approx 1.40 \text{ hr}.$$

Therefore 1.40 hours elapsed between the time of death and the time the body was found.

13 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 4$. The volume of solution in the tank at time t is $V(t) = 400 + 3t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned} x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\ &= \left(\frac{0.04 \text{ kg}}{1 \text{ L}}\right)\left(\frac{18 \text{ L}}{1 \text{ min}}\right) - \left(\frac{x(t) \text{ kg}}{V(t) \text{ L}}\right)\left(\frac{15 \text{ L}}{1 \text{ min}}\right) \\ &= 0.72 - \frac{15x(t)}{400 + 3t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{15x}{3t + 400} = 0.72.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{15}{3t + 400}\, dt\right) = e^{5\ln(3t+400)} = (3t + 400)^5$$

to obtain

$$(3t + 400)^5 x' + 15(3t + 400)^4 x = 0.72(3t + 400)^5,$$

which becomes

$$\left[(3t + 400)^5 x\right]' = 0.72(3t + 400)^5$$

and thus

$$(3t + 400)^5 x = 0.72 \int (3t + 400)^5\, dt = 0.72\left[\frac{1}{18}(3t + 400)^6\right] + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{3t + 400}{25} + \frac{c}{(3t + 400)^5}.$$

To find c we use the initial condition $x(0) = 4$, giving $c = -12(400^5)$, and so

$$x(t) = \frac{3t + 400}{25} - 12\left(\frac{400}{3t + 400}\right)^5.$$