Math 250 Exam #2 Key (Summer 2016)

1 Newton's Law of Cooling states that $T'(t) = k[T(t) - M]$, where M is the temperature of the oven. Here we have $T(0) = 70$, $T(0.5) = 120$, and $T(1) = 160$. Now,

$$T' = k(T - M) \;\Rightarrow\; \int \frac{dT}{T - M} = \int k\,dt \;\Rightarrow\; \ln|T - M| = kt + c \;\Rightarrow\; M - T = e^{kt+c},$$

and so

$$T(t) = M - Ce^{kt}.$$

From $T(0) = 70$ we obtain $70 = M - C$, so $C = M - 70$ and then

$$T(t) = M - (M - 70)e^{kt}.$$

From $T(0.5) = 120$ we obtain

$$120 = M - (M - 70)e^{0.5k} \;\Rightarrow\; e^{0.5k} = \frac{120 - M}{70 - M} \;\Rightarrow\; k = \ln\left(\frac{120 - M}{70 - M}\right)^2.$$

Thus

$$T(t) = M - (M - 70)\left(\frac{120 - M}{70 - M}\right)^{2t}$$

Now we use $T(1) = 160$ to get

$$160 = M - (70 - M)\left(\frac{120 - M}{70 - M}\right)^2,$$

which solves nicely to give $M = 320°\text{F}$.

2 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 5$. The volume of solution in the tank is $V(t) = 400 + 5t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned} x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\ &= \left(\frac{0.05\text{ kg}}{1\text{ L}}\right)\left(\frac{20\text{ L}}{1\text{ min}}\right) - \left(\frac{x(t)\text{ kg}}{V(t)\text{ L}}\right)\left(\frac{15\text{ L}}{1\text{ min}}\right) \\ &= 1 - \frac{15x(t)}{400 + 5t} = 1 - \frac{3x(t)}{80 + t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{3x}{t + 80} = 1.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{3}{t + 80}\,dt\right) = e^{3\ln(t+80)} = (t + 80)^3$$

to obtain

$$(t + 80)^3 x' + 3(t + 80)^2 x = (t + 80)^3,$$

which becomes

$$\left[(t + 80)^3 x\right]' = (t + 80)^3$$

and thus

$$(t+80)^3 x = \int (t+80)^3\,dt = \frac{1}{4}(t+80)^4 + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{t}{4} + \frac{c}{(t+80)^3} + 20.$$

To determine c we use the initial condition $x(0) = 5$, giving $c = -15(80^3)$, and so

$$x(t) = \frac{t}{4} - 15\left(\frac{80}{t+80}\right)^3 + 20.$$

The amount of sugar in the tank after 1 hour (60 minutes) is

$$x(60) = \frac{60}{4} - 15\left(\frac{80}{140}\right)^3 + 20 \approx 32.2 \text{ kg}.$$

3 Suppose c_1, c_2, c_3 are constants such that $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$. That is,

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

for all $x \in \mathbb{R}$, and hence

$$c_1 x^2 + c_2(6x^2 - 1) + c_3(2x^2 + 3) = 0$$

for all $x \in \mathbb{R}$. Rewrite this as

$$(c_1 + 6c_2 + 2c_3)x^2 + (-c_2 + 3c_3) = 0,$$

and note that if we let $c_2 = 3$ and $c_3 = 1$, then the constant term $-c_2 + 3c_3$ is eliminated. Now all we need do is set $c_1 = -20$ to also eliminate the x^2 term. That is, if we choose $c_1 = -20$, $c_2 = 3$, and $c_3 = 1$, then $c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$ is satisfied for all $x \in \mathbb{R}$. Therefore f, g, and h are linearly dependent on $(-\infty, \infty)$.

4 The auxiliary equation $r^2 - 10r + 25 = 0$ has double root 5, and so the general solution is

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}.$$

5 The auxiliary equation is $r^4 + r^3 + r^2 = 0$, or $r^2(r^2 + r + 1) = 0$, which has double root 0 and complex roots $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. The general solution is thus

$$y(x) = c_1 + c_2 x + e^{-x/2}\left(c_3 \cos\tfrac{\sqrt{3}}{2}x + c_4 \sin\tfrac{\sqrt{3}}{2}x\right).$$

6 Put equation in standard form: $y'' + 2t^{-1}y' - 6t^{-2}y = 0$, so $P(t) = 2/t$. We're given that $y_1(t) = t^2$ is a solution. From this we obtain

$$y_2(t) = y_1(t)\int \frac{e^{-\int P(t)\,dt}}{y_1^2(t)}dt = t^2 \int \frac{e^{-2\ln|t|}}{t^4}dt = t^2\int \frac{1}{t^6}dt = t^2\left(-\tfrac{1}{5}t^{-5} + c\right) = -\frac{1}{5t^3} + ct^2$$

for any $c \in \mathbb{R}$. If we let $c = 0$ then we get $y_2(t) = -1/5t^3$.

7a First consider $y'' + 2y' = 2t + 5$. Auxiliary equation is $r^2 + 2r = 0$, which has roots $r = -2, 0$. Now, the nonhomogeneity $f_1(t) = 2t + 5$ has the form $P_m(t)e^{\alpha t}$ with $m = 1$ and $\alpha = 0$, and since 0 is a root of the auxiliary equation we will need $s = 1$ in the form for the particular solution y_{p_1}. We have:

$$y_{p_1}(t) = t^s e^{\alpha t} \sum_{k=0}^{m} A_k t^k = t(A_0 + A_1 t) = At + Bt^2$$

(it's convenient to let $A = A_0$ and $B = A_1$). Thus $y'_{p_1}(t) = A + 2Bt$ and $y''_{p_1}(t) = 2B$. Putting all this into $y'' + 2y' = 2t + 5$ gives

$$2B + 2(A + 2Bt) = 2t + 5 \quad \Rightarrow \quad 4Bt + (2A + 2B) = 2t + 5,$$

so that $4B = 2$ and $2A + 2B = 5$, and finally $A = 2$ and $B = \frac{1}{2}$. Therefore $y_{p_1}(t) = 2t + \frac{1}{2}t^2$.

Next consider $y'' + 2y' = -e^{-2t}$. The nonhomogeneity $f_2(t) = -e^{-2t}$ has the form $P_m(t)e^{\alpha t}$ with $m = 0$ and $\alpha = -2$, and since -2 is a root of the auxiliary equation we will need $s = 1$ in the form for the particular solution y_{p_2}. We have

$$y_{p_2}(t) = t^s e^{\alpha t} \sum_{k=0}^{m} A_k t^k = tAe^{-2t},$$

so

$$y'_{p_2}(t) = (-2At + A)e^{-2t} \quad \text{and} \quad y''_{p_2}(t) = (4At - 4A)e^{-2t}.$$

Putting all this into $y'' + 2y' = -e^{-2t}$ and simplifying gives $-2Ae^{-2t} = -e^{-2t}$, and thus $A = \frac{1}{2}$. Therefore $y_{p_2}(t) = \frac{1}{2}te^{-2t}$.

By the Superposition Principle we conclude that

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) = 2t + \tfrac{1}{2}t^2 + \tfrac{1}{2}te^{-2t}$$

is a particular solution of the original equation.

7b General solution is

$$y(t) = c_1 + c_2 e^{-2t} + 2t + \tfrac{1}{2}t^2 + \tfrac{1}{2}te^{-2t}.$$

8 The nonhomogeneity is $f(t) = -2$, so has form $P_m(t)e^{\alpha t}$ with $m = 0$ and $\alpha = 0$. Auxiliary equation: $r^2 + 4 = 0$, which has roots $r = \pm 2i$. Thus $\alpha = 0$ is not a root of the auxiliary equation. By the Method of Undetermined Coefficients we have $y_p(t) = A$, which when put into the ODE easily gives $A = -\frac{1}{2}$, and so $y_p(t) = -\frac{1}{2}$. General solution is therefore

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \tfrac{1}{2}.$$

Now, from $y(\pi/8) = \frac{1}{2}$ we obtain $c_1 + c_2 = \sqrt{2}$, and from

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t.$$

and $y'(\pi/8) = 2$ we obtain $-c_1 + c_2 = \sqrt{2}$. Adding $c_1 + c_2 = \sqrt{2}$ and $-c_1 + c_2 = \sqrt{2}$ gives $2c_2 = 2\sqrt{2}$, or $c_2 = \sqrt{2}$, from which it follows that $c_1 = 0$. Therefore

$$y(t) = \sqrt{2} \sin 2t - \tfrac{1}{2}$$

is the solution to the IVP.

9 From the auxiliary equation $r^2 + 1 = 0$ we obtain $r = \pm i$, and so

$$y_1(t) = \cos t \quad \text{and} \quad y_2(t) = \sin t$$

are two linearly independent solutions to $y'' + y = 0$. The Wronskian of y_1 and y_2 is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = 1.$$

Now, with $a_2 = 1$ and $f(t) = \sec^2 t$, and making the substitution $u = \cos t$, we have

$$v_1(t) = \frac{1}{a_2}\int \frac{-y_2(t)f(t)}{W[y_1, y_2](t)}dt = -\int \frac{\sin t}{\cos^2 t}dt = \int \frac{1}{u^2}du = -\frac{1}{u} = -\sec t$$

and

$$v_2(t) = \frac{1}{a_2}\int \frac{y_1(t)f(t)}{W[y_1, y_2](t)}dt = \int \sec t\, dt = \ln|\sec t + \tan t|.$$

A particular solution to the ODE is

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = -\sec t \cos t + \sin t \cdot \ln|\sec t + \tan t| = \sin t \cdot \ln|\sec t + \tan t| - 1,$$

and so the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + \sin t \cdot \ln|\sec t + \tan t| - 1.$$

10 The IVT is

$$y'' + 10y' + 16y = 0, \quad y(0) = 1, \quad y'(0) = -12.$$

The auxiliary equation $r^2 + 10r + 16 = 0$ has roots -8 and -2, and so the general solution to the ODE is

$$y(t) = c_1 e^{-2t} + c_2 e^{-8t}.$$

With the initial conditions we find that $c_1 = -\frac{2}{3}$ and $c_2 = \frac{5}{3}$. The equation of motion is therefore

$$y(t) = -\tfrac{2}{3}e^{-2t} + \tfrac{5}{3}e^{-8t}.$$