Math 250 Exam #1 Key (Summer 2016)

1 The initial-value problem

$$(9-y^2)y' = x^2, \quad y(x_0) = y_0,$$

will have a unique solution if

$$f(x,y) = \frac{x^2}{9-y^2} \quad \text{and} \quad f_y(x,y) = \frac{2x^2y}{(9-y^2)^2}$$

are continuous on some open rectangle containing (x_0, y_0). This will be the case for any $(x_0, y_0) \in \mathbb{R}^2$ such that $y_0 \neq \pm 3$. That is, if (x_0, y_0) lies off the horizontal lines $y = 3$ and $y = -3$ in the xy-plane, then the IVP will have a unique solution.

2 Let $V(t)$ be the volume of the water in the tank at time t. Thus we have

$$V'(t) = -cA\sqrt{2gh(t)},$$

where $h(t)$ is the height of the water at time t, and the negative sign is introduced to indicate the volume of the water is decreasing. However we also have $V(t) = 100h(t)$, and so $V'(t) = 100h'(t)$. This gives

$$100h'(t) = -cA\sqrt{2gh(t)} \quad \Rightarrow \quad h'(t) = -\frac{cA\sqrt{2gh(t)}}{100}.$$

Letting $g = 32$ and $A = \pi(1/6)^2 = \pi/36$ (since 2 inches is 1/6 ft) yields

$$h'(t) = -\frac{c\pi}{450}\sqrt{h(t)} \quad \text{or} \quad \frac{dh}{dt} = -\frac{c\pi}{450}\sqrt{h}.$$

3 We have $x^2y' = y(1-x)$, which by separation of variables yields

$$\int \frac{1}{y}dy = \int \frac{1-x}{x^2}dx \quad \Rightarrow \quad \ln|y| = -\frac{1}{x} - \ln|x| + c \quad \Rightarrow \quad \ln|xy| = c - \frac{1}{x}.$$

With the given initial condition, $y(-1) = -1$, we have $\ln|(-1)(-1)| = c - (-1)$, or $c = -1$; and since the initial condition has $x < 0$ and $y < 0$, it follows that $|x| = -x$ and $|y| = -y$, and thus

$$|xy| = |x||y| = (-x)(-y) = xy.$$

The solution is therefore $\ln(xy) = -1 - 1/x$.

4 Rewrite the equation as

$$2y\cos^3(3x)\frac{dy}{dx} = -\sin(3x).$$

By separation of variables we get

$$\int 2y\,dy = -\int \frac{\sin 3x}{\cos^3 3x}dx.$$

For the integral at right, perform the substitution $u = \cos 3x$, so

$$y^2 = \frac{1}{3}\int \frac{1}{u^3}du = -\frac{1}{6u^2} + c = -\frac{1}{6\cos^2 3x} + c,$$

and hence

$$y^2 = -\frac{1}{6}\sec^2 3x + c.$$

5 Get the equation in the standard form:

$$\frac{dP}{dt} + (2t-1)P = 4t - 2.$$

Multiply by the integrating factor

$$\mu(t) = e^{\int (2t-1)dt} = e^{t^2-t}$$

to get

$$e^{t^2-t}\frac{dP}{dt} + (2t-1)e^{t^2-t}P = (4t-2)e^{t^2-t},$$

or

$$\left(e^{t^2-t}P\right)' = (4t-2)e^{t^2-t}.$$

Integrate both sides, performing the substitution $u = t^2 - t$ along the way:

$$e^{t^2-t}P = \int (4t-2)e^{t^2-t}dt = 2\int e^u\,du = 2e^u + c = 2e^{t^2-t} + c.$$

Therefore

$$P(t) = 2 + ce^{t-t^2}.$$

6 We find an integrating factor:

$$\mu(x) = e^{\int 4x\,dx} = e^{2x^2},$$

Multiply ODE by $\mu(x)$ to get:

$$e^{2x^2}y' + 4xe^{2x^2}y = x^3e^{3x^2} \quad\Rightarrow\quad (e^{2x^2})' = x^3e^{3x^2} \quad\Rightarrow\quad e^{2x^2}y = \int x^3e^{3x^2}\,dx + c$$

The substitution $u = 3x^2$ then gives:

$$e^{2x^2}y = \frac{1}{18}\int ue^u\,du + c = \frac{1}{18}(ue^u - e^u) + c = \frac{1}{18}(3x^2e^{3x^2} - e^{3x^2}) + c.$$

Therefore

$$y = \left(\frac{1}{6}x^2 - \frac{1}{18}\right)e^{x^2} + ce^{-2x^2}.$$

Initial condition: $y = -1$ when $x = 0$, so $-1 = -\frac{1}{18} + c$, giving $c = -\frac{17}{18}$, and finally

$$y = \left(\frac{1}{6}x^2 - \frac{1}{18}\right)e^{x^2} - \frac{17}{18}e^{-2x^2}.$$

7 We find a function F such that $F_x(x,y) = e^x + y$ and $F_y(x,y) = 2 + x + ye^y$. Now,

$$F(x,y) = \int (e^x + y)dx = e^x + xy + g(y)$$

for arbitrary differentiable function g. Then

$$2 + x + ye^y = F_y(x,y) = x + g'(y) \quad\Rightarrow\quad g'(y) = 2 + ye^y \quad\Rightarrow\quad g(y) = 2y + ye^y - e^y,$$

so

$$F(x,y) = e^x + xy + 2y + ye^y - e^y.$$

Solution to ODE is $F(x,y) = c$; that is,

$$e^x + xy + 2y + ye^y - e^y = c.$$

Initial condition gives $y = 1$ when $x = 0$, so $1 + 0 + 2 + e - e = c$, or $c = 3$, and therefore the solution to the IVP is

$$e^x + (y-1)e^y + (x+2)y = 3.$$

8 Let $v = x + y$, so $v' = 1 + y'$, and thus $y' = v' - 1$. Substituting this into the ODE gives $v' - 1 = \tan^2 v$, and thus $v' = \sec^2 v$. This is a separable equation:

$$\int \cos^2 v \, dv = \int dx.$$

Use the identity $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ for the win:

$$\frac{\sin 2v}{4} + \frac{v}{2} + c = x \quad\Rightarrow\quad x = \frac{\sin(2x+2y)}{4} + \frac{x+y}{2} + c \quad\Rightarrow\quad x = \frac{\sin(2x+2y)}{2} + y + c.$$

The supplied identity $\sin 2\theta = 2\sin\theta\cos\theta$ may be used to get

$$x = \sin(x+y)\cos(x+y) + y + c.$$

9 Rewrite equation thus: $y' + y = xy^4$. This is Bernoulli with $n = 4$, $P(x) = 1$, and $Q(x) = x$. Letting $v = y^{1-n} = y^{-3}$, we obtain the linear equation

$$v' - 3v = -3x.$$

Multiplying by the integrating factor $\mu(x) = e^{-3x}$ yields

$$e^{-3x}v' - 3e^{-3x}v = -3xe^{-3x} \quad\Rightarrow\quad (e^{-3x}v)' = -3xe^{-3x} \quad\Rightarrow\quad e^{-3x}v = -3\int xe^{-3x}dx,$$

whence

$$e^{-3x}v = -3\left[-\tfrac{1}{3}xe^{-3x} - \tfrac{1}{9}e^{-3x} + c\right] = xe^{-3x} + \tfrac{1}{3}e^{-3x} + c \quad\Rightarrow\quad v(x) = x + \tfrac{1}{3} + ce^{3x},$$

and finally

$$y^{-3} = x + \frac{1}{3} + ce^{3x} \quad\text{ or }\quad y = \sqrt[3]{\frac{3}{3x + 1 + ce^{3x}}}.$$

Also $y \equiv 0$ is a solution.