Math 250 Exam #4 Key (Summer 2015)

1 We have

$$\mathcal{L}[f] = \int_0^2 e^{-st}dt + \int_2^\infty te^{-st}dt = -\frac{1}{s}(e^{-2s}-1) + \left(\frac{2}{s}+\frac{1}{s^2}\right)e^{-2s} = \frac{1}{s} + \left(\frac{1}{s}+\frac{1}{s^2}\right)e^{-2s}.$$

2 Applying linearity properties,

$$\begin{aligned}\mathcal{L}[f] &= \mathcal{L}\left[8t^3-12t^2+6t-1\right] = 8\mathcal{L}\left[t^3\right] - 12\mathcal{L}\left[t^2\right] + 6\mathcal{L}[t] - \mathcal{L}[1] \\ &= 8\cdot\frac{3!}{s^4} - 12\cdot\frac{2!}{s^3} + 6\cdot\frac{1!}{s^2} - \frac{0!}{s} = \frac{48}{s^4} - \frac{24}{s^3} + \frac{6}{s^2} - \frac{1}{s}.\end{aligned}$$

3 Applying partial fraction decomposition and linearity properties,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s+1}{s^2-4s}\right] &= \mathcal{L}^{-1}\left[\frac{s+1}{s(s-4)}\right] = \mathcal{L}^{-1}\left[\frac{-1/4}{s} + \frac{5/4}{s-4}\right] \\ &= -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] = -\frac{1}{4} + \frac{5}{4}e^{4t}.\end{aligned}$$

4 Let $Y = \mathcal{L}[y]$, so the Laplace transform of the ODE is

$$\mathcal{L}[y'] - \mathcal{L}[y] = 2\mathcal{L}[\cos 5t] \;\Rightarrow\; sY - y(0) - Y = \frac{2s}{s^2+25} \;\Rightarrow\; Y = \frac{2s}{(s-1)(s^2+25)},$$

and hence

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left[\frac{2s}{(s-1)(s^2+25)}\right] = \mathcal{L}^{-1}\left[\frac{\frac{1}{13}}{s-1} + \frac{\frac{25}{13}-\frac{1}{13}s}{s^2+25}\right] \\ &= \frac{1}{13}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{13}\left(\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right] - 5\mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right]\right) \\ &= \frac{1}{13}e^t - \frac{1}{13}\big(\cos 5t - 5\sin 5t\big) = \frac{e^t - \cos 5t + 5\sin 5t}{13}.\end{aligned}$$

5a Apply partial fraction decomposition:

$$\mathcal{L}^{-1}\left[\frac{5s}{(s-2)^2}\right] = 5\mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{2}{(s-2)^2}\right] = 5\big(e^{2t} + 2te^{2t}\big) = (10t+5)e^{2t}.$$

5b Apply $\mathcal{L}[g(t)u(t-a)] = e^{-as}\mathcal{L}[g(t+a)]$ with $g(t) = 3t+1$ and $a = 1$:

$$\mathcal{L}[(3t+1)u(t-1)] = e^{-s}\mathcal{L}[3(t+1)+1] = e^{-s}\mathcal{L}[3t+4] = e^{-s}\left(\frac{3}{s^2} + \frac{4}{s}\right).$$

5c Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right],$$

so that

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s}}{s^2+4} = e^{-\pi s}\mathcal{L}\left[\tfrac{1}{2}\sin 2t\right].$$

Now apply the formula

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\mathcal{L}[f(t)] \tag{1}$$

with $f(t) = \frac{1}{2}\sin 2t$ and $a = \pi$ to obtain

$$\mathcal{L}\left[\tfrac{1}{2}\sin 2(t-\pi)u(t-\pi)\right] = e^{-\pi s}\mathcal{L}\left[\tfrac{1}{2}\sin 2t\right] = \frac{e^{-\pi s}}{s^2+4},$$

and therefore

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right] = \tfrac{1}{2}\sin 2(t-\pi)u(t-\pi) = \tfrac{1}{2}\sin(2t)u(t-\pi).$$

6 We have $y' + 2y = t - tu(t-1)$. Let $Y = \mathcal{L}[y]$, which is a function of s. Taking the Laplace transform of the equation yields

$$[sY - y(0)] + 2Y = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right),$$

so that

$$Y = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{s+1}{s^2(s+2)}\right),$$

and thus

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2(s+2)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}(s+1)}{s^2(s+2)}\right] \tag{2}$$

By partial fraction decomposition we have

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2},$$

so that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+2)}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = -\tfrac{1}{4} + \tfrac{1}{2}t + \tfrac{1}{4}e^{-2t}.$$

The other term in (2) is handled similarly, giving

$$y(t) = -\tfrac{1}{4} + \tfrac{1}{2}t + \tfrac{1}{4}e^{-2t} - \tfrac{1}{4}u(t-1) - \tfrac{1}{2}(t-1)u(t-1) + \tfrac{1}{4}e^{-2(t-1)}u(t-1).$$

7 Taking the Laplace transform of the equation gives

$$\begin{aligned}\mathcal{L}[f(t)] &= \mathcal{L}\left[te^t\right] + \mathcal{L}\left[\int_0^t \tau f(t-\tau)d\tau\right] \\ &= \mathcal{L}\left[te^t\right] + \mathcal{L}[t]\mathcal{L}[f(t)] = \frac{1}{(s-1)^2} + \frac{1}{s^2}\mathcal{L}[f(t)],\end{aligned}$$

and hence

$$\mathcal{L}[f(t)] = \frac{s^2}{(s+1)(s-1)^3} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}.$$

Applying the partial fraction decomposition method, we obtain

$$\mathcal{L}[f(t)] = \frac{-1/8}{s+1} + \frac{1/8}{s-1} + \frac{3/4}{(s-1)^2} + \frac{1/2}{(s-1)^3},$$

so that

$$f(t) = \mathcal{L}^{-1}\left[\frac{-1/8}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1/8}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{3/4}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1/2}{(s-1)^3}\right],$$

and finally

$$f(t) = -\tfrac{1}{8}e^{-t} + \tfrac{1}{8}e^{t} + \tfrac{3}{4}te^{t} + \tfrac{1}{4}t^2e^{t}.$$

8 We take the Laplace transform of the ODE:

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] = \mathcal{L}[\delta(t-1)].$$

Now, letting $Y = \mathcal{L}[y]$ we obtain

$$[s^2Y - sy(0) - y'(0)] + [sY - y(0)] = e^{-s} \quad \Rightarrow \quad (s^2Y - 1) + sY = e^{-s}$$

whence

$$Y = \frac{1 + e^{-s}}{s^2 + s}$$

and so

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+s}\right] + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+s}\right]. \tag{3}$$

Partial fractions decomposition gives

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}. \tag{4}$$

Using (4), and also the formula (1) with $f(t) = 1 - e^{-t}$ and $a = 1$, we get

$$\mathcal{L}\big[\big(1 - e^{-(t-1)}\big)u(t-1)\big] = e^{-s}\mathcal{L}\big[1 - e^{-t}\big] = \frac{e^{-s}}{s^2+s},$$

and therefore (3) becomes

$$y(t) = 1 - e^{-t} + (1 - e^{1-t})u(t-1).$$