Math 250 Exam #3 Key (Summer 2015)

1 The auxiliary equation $r^2-2r+1=0$ has double root 1, so the corresponding homogeneous equation $y''-2y'+1=0$ has linearly independent solutions $y_1(t)=e^t$ and $y_2(t)=te^t$, and thus

$$y_1'(t)=e^t \quad \text{and} \quad y_2'(t)=(1+t)e^t.$$

Now, since $a_2=1$ and $f(t)=(1+t^2)^{-1}e^t$, we have

$$v_1(t)=\int \frac{-te^t\cdot(1+t^2)^{-1}e^t}{e^{2t}}dt=-\int\frac{t}{1+t^2}dt=-\frac{1}{2}\ln(1+t^2),$$

and

$$v_2(t)=\int\frac{e^t\cdot(1+t^2)^{-1}e^t}{e^{2t}}dt=\int\frac{1}{1+t^2}dt=\tan^{-1}(t).$$

Hence a particular solution to the ODE is

$$y_p(t)=v_1(t)y_1(t)+v_2(t)y_2(t)=-\tfrac{1}{2}e^t\ln(1+t^2)+te^t\tan^{-1}(t).$$

General solution is therefore

$$y(t)=\left[-\tfrac{1}{2}\ln(1+t^2)+t\tan^{-1}(t)+c_1t+c_2\right]e^t.$$

2a By Hooke's Law $F=ky$ we calculate the spring constant k as $k=F/y=64/0.32=200$ lb/ft. Mass m and weight W are related by $W=mg$, and so mass is $m=W/g=64/32=2$ slugs. There is no frictional force mentioned, so the system is assumed to be undamped. The IVP that models the mass-spring system is

$$2y''+200y=0, \quad y(0)=-\tfrac{2}{3}\text{ ft}, \quad y'(0)=5\text{ ft/s}.$$

Note it is necessary to convert inches to feet! The auxiliary equation $2r^2+200=0$ has solutions $r=\pm10i$, and so the solution to the ODE is

$$y(t)=c_1\cos 10t+c_2\sin 10t.$$

With the initial conditions we determine the solution to the IVP to be

$$y(t)=-\tfrac{2}{3}\cos 10t+\tfrac{1}{2}\sin 10t.$$

2b From

$$y'(t)=\tfrac{20}{3}\sin 10t+5\cos 10t \quad \text{and} \quad y''(t)=\tfrac{200}{3}\cos 10t-50\sin 10t$$

we reckon

$$y(3)=-0.60\text{ ft}, \quad y'(3)=-5.82\text{ ft/s}, \quad y''(3)=59.69\text{ ft/s}^2.$$

2c The period is $P=\pi/5$ seconds and the amplitude is

$$A=\sqrt{\left(-\tfrac{2}{3}\right)^2+\left(\tfrac{1}{2}\right)^2}=\tfrac{5}{6}\text{ ft}.$$

2d We must first find the times t for which $y(t)=0$. We have

$$y(t)=0 \;\Rightarrow\; \tfrac{1}{2}\sin 10t=\tfrac{2}{3}\cos 10t \;\Rightarrow\; \tan 10t=\tfrac{4}{3} \;\Rightarrow\; t=\arctan\left(\tfrac{4}{3}\right).$$

Thus the object is at the equilibrium position at time $t = \arctan\left(\frac{4}{3}\right) + \frac{\pi}{10}n$, where $n \geq 0$ is any integer. When $n = 0$ we obtain $\arctan\left(\frac{4}{3}\right) \approx 0.0927$ second as the first time. The velocity at this time is

$$y'(0.0927) = \tfrac{20}{3}\sin(0.927) + 5\cos(0.927) \approx 8.33 \text{ ft/s}.$$

The velocity at successive times will alternate between -8.33 ft/s and 8.33 ft/s.

3 We have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 5\sum_{n=0}^{\infty} c_n x^{n+2} &= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - 5\sum_{n=2}^{\infty} c_{n-2}x^n \\ &= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=2}^{\infty} 5c_{n-2}x^n \\ &= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)c_{n+2} - 5c_{n-2}\right]x^n.\end{aligned}$$

4 Substituting

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

into the ODE gives

$$(2x-4)\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0,$$

which with reindexing becomes

$$\sum_{n=0}^{\infty}\left[(2n+1)c_n - 4(n+1)c_{n+1}\right]x^n = 0.$$

This implies that

$$(2n+1)c_n - 4(n+1)c_{n+1} = 0$$

for all $n \geq 0$, so in particular

$$c_1 = \frac{1}{4}c_0, \quad c_2 = \frac{3}{4^2 \cdot 2!}c_0, \quad c_3 = \frac{3 \cdot 5}{4^3 \cdot 3!}c_0, \quad c_4 = \frac{3 \cdot 5 \cdot 7}{4^4 \cdot 4!}c_0,$$

and in general

$$c_n = \frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{4^n \cdot n!}c_0,$$

so that

$$y = c_0\sum_{n=0}^{\infty}\frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{4^n \cdot n!}x^n = c_0\sum_{n=0}^{\infty}\frac{2(2n-1)!}{(n-1)!n!}\left(\frac{x}{8}\right)^n.$$

5 Substituting $y=\sum_{n=0}^{\infty} c_n x^n$ into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + 8\sum_{n=0}^{\infty} c_n x^n = 0,$$

whence comes

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=0}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} 8c_n x^n = 0,$$

and then

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)c_{n+2} - 2nc_n + 8c_n\right]x^n = 0.$$

This implies that

$$(n+1)(n+2)c_{n+2} - 2nc_n + 8c_n = 0,$$

for all $n \ge 0$, and hence

$$c_{n+2} = \frac{2n-8}{(n+1)(n+2)}c_n.$$

We now calculate

$$c_2 = -4c_0, \quad c_3 = \frac{-6}{3!}c_1, \quad c_4 = \frac{4}{3}c_0, \quad c_5 = \frac{(-6)(-2)}{5!}c_1, \quad c_6 = 0, \quad c_7 = \frac{(-6)(-2)(2)}{7!}c_1,$$

$$c_8 = 0, \quad c_9 = \frac{(-6)(-2)(2)(6)}{9!}, \quad c_{10} = 0,$$

and in general

$$c_{2n+1} = \frac{(-6)(-2)(2)\cdots(4n-10)}{(2n+1)!}c_1$$

for $n \ge 0$, and $c_{2n} = 0$ for $n \ge 3$. Now, since

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n}x^{2n} + \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1},$$

we conclude that

$$y = c_0\left(1 - 4x^2 + \frac{4}{3}x^4\right) + c_1\left(x + \sum_{n=1}^{\infty}\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n+1)!}x^{2n+1}\right).$$

This along with the initial condition $y(0) = 3$ yields $c_0 = 3$. From

$$y' = -8c_0 x + \frac{16}{3}c_0 x^3 + c_1\left(1 + \sum_{n=1}^{\infty}\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n)!}x^{2n}\right)$$

and the initial condition $y'(0) = 0$ we get $c_1 = 0$. Therefore

$$y = 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 3 - 12x^2 + 4x^4$$

is the solution to the IVP.