Math 250 Exam #2 Key (Summer 2015)

1 Newton's Law of Cooling states that $T'(t) = k[T(t) - T_a]$. Here we have $T_a = 5$, $T(1) = 55$, and $T(5) = 30$. Now, noting that $T(t) \geq 5$ for all $t \geq 0$,

$$T' = k(T-5) \;\Rightarrow\; \int \frac{1}{T-5}dT = \int k\,dt \;\Rightarrow\; \ln|T-5| = kt + c \;\Rightarrow\; T - 5 = e^{kt+c},$$

and so

$$T(t) = 5 + Ce^{kt}.$$

From $T(1) = 55$ we obtain

$$55 = 5 + Ce^k \;\Rightarrow\; Ce^k = 50 \;\Rightarrow\; C = 50e^{-k},$$

and so

$$T(t) = 5 + 50e^{-k}e^{kt} = 5 + 50e^{k(t-1)}.$$

From $T(5) = 30$ we obtain

$$30 = 5 + 50e^{4k} \;\Rightarrow\; e^{4k} = \frac{1}{2} \;\Rightarrow\; 4k = \ln\left(\frac{1}{2}\right) \;\Rightarrow\; k = \frac{1}{4}\ln\left(\frac{1}{2}\right) \approx -0.173.$$

Thus

$$T(t) = 5 + 50e^{(t-1)\ln(0.5^{1/4})} = 5 + 50\left(\frac{1}{2}\right)^{(t-1)/4} = 5 + 50\sqrt[4]{\frac{1}{2^{t-1}}}.$$

The temperature in the house is given by

$$T(0) = 5 + 50\sqrt[4]{\frac{1}{2^{0-1}}} \approx 64.5°\text{F}.$$

2 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 5$. The volume of solution in the tank is $V(t) = 400 + 5t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned} x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\ &= \left(\frac{0.05\text{ kg}}{1\text{ L}}\right)\left(\frac{20\text{ L}}{1\text{ min}}\right) - \left(\frac{x(t)\text{ kg}}{V(t)\text{ L}}\right)\left(\frac{15\text{ L}}{1\text{ min}}\right) \\ &= 1 - \frac{15x(t)}{400+5t} = 1 - \frac{3x(t)}{80+t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{3x}{t+80} = 1.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{3}{t+80}\,dt\right) = e^{3\ln(t+80)} = (t+80)^3$$

to obtain

$$(t+80)^3x' + 3(t+80)^2x = (t+80)^3,$$

which becomes

$$\left[(t+80)^3x\right]' = (t+80)^3$$

and thus

$$(t+80)^3 x = \int (t+80)^3\,dt = \frac{1}{4}(t+80)^4 + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{t}{4} + \frac{c}{(t+80)^3} + 20.$$

To determine c we use the initial condition $x(0) = 5$, giving $c = -15(80^3)$, and so

$$x(t) = \frac{t}{4} - 15\left(\frac{80}{t+80}\right)^3 + 20.$$

The amount of sugar in the tank after 30 minutes is

$$x(30) = \frac{30}{4} - 15\left(\frac{80}{110}\right)^3 + 20 \approx 21.7 \text{ kg}.$$

3 Suppose c_1, c_2, c_3 are constants such that $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$. That is,

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

for all $x \in \mathbb{R}$, and hence

$$c_1 x + c_2(x-1) + c_3(x+3) = 0$$

for all $x \in \mathbb{R}$. If $c_2 = 3$ and $c_3 = 1$ then we get

$$c_1 x + 3(x-1) + (x+3) = 0,$$

and hence $c_1 x + 4x = 0$. This last equation is satisfied on $(-\infty, \infty)$ if we let $c_1 = -4$. That is, $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$ is possible if we choose $c_1 = -4$, $c_2 = 3$, and $c_3 = 1$. Since $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$ admits a solution other than $c_1 = c_2 = c_3 = 0$, we conclude that f, g, and h are linearly dependent on $(-\infty, \infty)$.

4 We have $y(x) = c_1 + c_2 x^2$ and $y'(x) = 2c_2 x$. From $y(0) = 1$ we get $1 = c_1 + c_2(0^2) = c_1$, or $c_1 = 1$; and from $y'(1) = 6$ we get $6 = 2c_2(1)$, or $c_2 = 3$. Therefore a solution to the boundary value problem is $y = 1 + 3x^2$.

5 A second solution y_2 will be of the form

$$y_2(x) = u(x)y_1(x) = xu(x)\sin(\ln x),$$

which we'll write simply as $y = xu\sin(\ln x)$. Now,

$$y' = xu \cdot \frac{\cos(\ln x)}{x} + (u + xu')\cdot\sin(\ln x) = u\cos(\ln x) + u\sin(\ln x) + xu'\sin(\ln x),$$

and

$$y'' = \frac{u\cos(\ln x) - u\sin(\ln x)}{x} + 2u'\cos(\ln x) + 2u'\sin(\ln x) + xu''\sin(\ln x).$$

Oh dear. Oh deary me. Substituting all this into $x^2y'' - xy' + 2y = 0$ gives

$$x\big[u\cos(\ln x) - u\sin(\ln x)\big] + x^2\big[2u'\cos(\ln x) + 2u'\sin(\ln x) + xu''\sin(\ln x)\big]$$

$$- x\big[u\cos(\ln x) + u\sin(\ln x) + xu'\sin(\ln x)\big] + 2xu\sin(\ln x) = 0,$$

whence

$$x^3\sin(\ln x)u'' + x^2\big[\sin(\ln x) + 2\cos(\ln x)\big]u' = 0.$$

Letting $w = u'$ and dividing out $x^3\sin(\ln x)$ gives

$$w' + \frac{\sin(\ln x) + 2\cos(\ln x)}{x\sin(\ln x)}w = 0.$$

Another way to write this is

$$w' + \left(\frac{1}{x} + \frac{2\cos(\ln x)}{x\sin(\ln x)}\right)w = 0.$$

This is separable:

$$-\int\frac{1}{w}dw = \int\left(\frac{1}{x} + \frac{2\cos(\ln x)}{x\sin(\ln x)}\right)dx = \ln|x| + 2\int\frac{\cos(\ln x)}{x\sin(\ln x)}dx. \tag{1}$$

Making the substitution $\alpha = \ln x$, and then the substitution $\beta = \sin\alpha$, we have

$$\int\frac{\cos(\ln x)}{x\sin(\ln x)}dx = \int\frac{\cos\alpha}{\sin\alpha}d\alpha = \int\frac{1}{\beta}d\beta = \ln|\beta| = \ln|\sin\alpha| = \ln|\sin(\ln x)|.$$

Now (1) becomes

$$-\ln|u'| = -\ln|w| = \ln(x) + 2\ln|\sin(\ln x)| + c = \ln\big(x\sin^2(\ln x)\big) + c,$$

where $|x| = x$ since it's clear early on that we must have $x > 0$. Choosing $c = 0$, we then get

$$\frac{1}{|u'|} = x\sin^2(\ln x) \quad\Rightarrow\quad |u'| = \frac{1}{x\sin^2(\ln x)} \quad\Rightarrow\quad u'(x) = \pm\frac{\csc^2(\ln x)}{x}.$$

We can choose $u'(x) = -x^{-1}\csc^2(\ln x)$, so, making the substitution $\alpha = \ln x$ again, we have

$$u(x) = -\int\frac{\csc^2(\ln x)}{x}dx = -\int\csc^2\alpha\,d\alpha = \cot(\alpha) + c = \cot(\ln x) + c.$$

Again choosing $c = 0$, we obtain $u(x) = \cot(\ln x)$, and so

$$y_2(x) = x\cot(\ln x)\sin(\ln x) = x\cos(\ln x)$$

is a second solution to the ODE. Deary, deary me—we shan't have this one on an exam again.

6 The auxiliary equation $2r^2 + 2r + 1 = 0$ has roots $-\frac{1}{2} \pm \frac{1}{2}i$, and so the general solution is

$$y(x) = e^{-x/2}\big(c_1\cos(x/2) + c_2\sin(x/2)\big).$$

This makes up for Problem (5) a little bit.

7 The auxiliary equation $r^3 - 6r^2 + 12r - 8 = 0$. It can be seen by inspection that $r = 2$ is a root. With a little synthetic division,

$$\begin{array}{r|rrr|r} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

we obtain the factorization $(r-2)(r^2-4r+4)=0$, and thus $(r-2)(r-2)(r-2)=0$. That is, 2 is a root of the auxiliary equation with multiplicity 3. The general solution is thus

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} = (c_1 + c_2 x + c_3 x^2)e^{2x}.$$

8 The given general solution results from an auxiliary equation having root 0 with multiplicity 2, and root 8 with multiplicity 1. That is, $r^2(r-8)=0$, or $r^3-8r^2=0$. A suitable ODE is

$$y''' - 8y'' = 0.$$