Math 250 Exam #1 Key (Summer 2015)

1 Substitute $\varphi(x) = e^{mx}$ for y to obtain

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = 0 \;\Rightarrow\; 2m^2 + 7m - 4 = 0 \;\Rightarrow\; (2m-1)(m+4) = 0,$$

so $m \in \{1/2, -4\}$, and therefore $\varphi_1(x) = e^{x/2}$ and $\varphi_2(x) = e^{-4x}$ are solutions to the ODE. (Note that $y \equiv 0$ is a solution also.)

2 Given $y = \sin x$,

$$y' = \sqrt{1-y^2} \;\Leftrightarrow\; \cos x = \sqrt{1-\sin^2 x} \;\Leftrightarrow\; \cos x = \sqrt{\cos^2 x} \;\Leftrightarrow\; \cos x = |\cos x|.$$

Thus we must have $\cos x \geq 0$, and since we generally take solutions to differential equations to be defined on *open* intervals we must have $x \in (-\pi/2 + 2\pi n, \pi/2 + 2\pi n)$ for any $n \in \mathbb{Z}$. So we may let $I = (-\pi/2, \pi/2)$.

3 Rewrite equation as $y' = x^2/(1+y^3)$. According to the Existence-Uniqueness Theorem the IVP

$$y' = \frac{x^2}{1+y^3}, \quad y(x_0) = y_0,$$

has a unique solution if

$$f(x,y) = \frac{x^2}{1+y^3} \quad \text{and} \quad f_y(x,y) = -\frac{3x^2y^2}{(1+y^3)^2}$$

are both continuous on some open set in $\mathbb{R}^2$ containing (x_0, y_0). This will be the case for any $(x_0, y_0) \in \mathbb{R}^2$ with $y_0 \neq -1$.

4 Here $x'(t)$ denotes the rate of change of the drug's amount in the bloodstream with respect to time t, and so if we let k be the constant of proportionality we have $x'(t) = r - kx(t)$.

5 Rewriting the equation as

$$y' = -y^2 e^{\cos x} \sin x$$

shows it to be separable. We obtain

$$-\int \frac{1}{y^2}dy = \int e^{\cos x} \sin x\, dx.$$

Let $u = \cos x$ in the integral at right, so

$$\frac{1}{y} = -\int e^u du = -e^u + c = -e^{\cos x} + c$$

for arbitrary constant c. That is,

$$y = \frac{1}{c - e^{\cos x}}.$$

6 Writing the equation as $y' = 2\cos x\sqrt{y+1}$, we see the equation is separable. We get
$$\int \frac{1}{\sqrt{y+1}}\,dy = \int 2\cos x\,dx.$$
This integrates easily to give
$$2\sqrt{y+1} = 2\sin x + c.$$
Now, $y(\pi) = 0$ implies that
$$2\sqrt{0+1} = 2\sin\pi + c,$$
or $c = 2$. The (implicit) solution to the IVP is thus
$$2\sqrt{y+1} = 2\sin x + 2,$$
or $y = (\sin x + 1)^2 - 1$.

7 The equation may be written as
$$y' + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x,$$
which is the standard form for a 1st-order linear ODE. An integrating factor is
$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3.$$
Multiplying the ODE by x^3 gives
$$x^3y' + 3x^2y = x\sin x - 3x^4,$$
which becomes $(x^3y)' = x\sin x - 3x^4$ and thus
$$x^3y = \int x\sin x\,dx - \frac{3}{5}x^5 + c.$$
Integration by parts gives
$$\int x\sin x\,dx = \sin x - x\cos x,$$
so that $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + c$ and therefore
$$y(x) = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + c\right).$$
is the general solution.

8 The equation may be written as
$$x' + \frac{3}{t}x = t^2\ln t + \frac{1}{t^2},$$
which is the standard form for a 1st-order linear ODE. An integrating factor is
$$\mu(x) = e^{\int 3/t\,dt} = t^3.$$
Multiplying the ODE by t^3 gives $t^3x' + 3t^2x = t^5\ln t + t$, which becomes $(t^3x)' = t^5\ln t + t$ and thus
$$t^3x = \int t^5\ln t\,dt + \frac{1}{2}t^2 + c.$$

By integration by parts we find that

$$\int t^5 \ln t \, dt = \frac{1}{6} t^6 \ln t - \int \frac{1}{6} t^5 \, dt = \frac{t^6}{6} \ln t - \frac{t^6}{36},$$

and so we have

$$t^3 x = \frac{t^6}{6} \ln t - \frac{t^6}{36} + \frac{1}{2} t^2 + c.$$

Letting $t = 1$ and $x = 0$ (the initial condition) gives $0 = -\frac{1}{36} + \frac{1}{2} + c$, so that $c = -\frac{17}{36}$ and we obtain

$$x(t) = \frac{1}{6} t^3 \left(\ln t - \frac{1}{6} \right) + \frac{1}{2t} - \frac{17}{36t^3}$$

as the solution to the IVP.

9 We have

$$M(x,y) = x - y^3 + y^2 \sin x \quad \text{and} \quad N(x,y) = -3xy^2 - 2y \cos x.$$

Since the equation is exact there exists a function $F(x,y)$ such that $F_x = M$ and $F_y = N$; that is,

$$F_x(x,y) = x - y^3 + y^2 \sin x \quad \text{and} \quad F_y(x,y) = -3xy^2 - 2y \cos x. \tag{1}$$

Integrate the first equation in (1) with respect to x:

$$F(x,y) = \int (x - y^3 + y^2 \sin x) dx + g(y) = \tfrac{1}{2} x^2 - xy^3 - y^2 \cos x + g(y). \tag{2}$$

Differentiating this with respect to y yields

$$F_y(x,y) = -3xy^2 - 2y \cos x + g'(y),$$

and so from the second equation in (1) we obtain

$$-3xy^2 - 2y \cos x + g'(y) = -3xy^2 - 2y \cos x,$$

or simply $g'(y) = 0$. Hence $g(y) = c_1$ for some arbitrary constant c_1, and so (2) becomes

$$F(x,y) = \tfrac{1}{2} x^2 - xy^3 - y^2 \cos x + c_1.$$

The general implicit solution to the ODE is therefore

$$\tfrac{1}{2} x^2 - xy^3 - y^2 \cos x + c_1 = c_2$$

for arbitrary c_2, which we can write simply as

$$\tfrac{1}{2} x^2 - xy^3 - y^2 \cos x = c$$

by consolidating the arbitrary constants c_1 and c_2.

10 Rewrite ODE as

$$y' = -\frac{x^2 + y^2}{2xy} = -\frac{1 + (y/x)^2}{2(y/x)}.$$

Let $v = y/x$, so $y = xv$ and we get $y' = xv' + v$. ODE then becomes

$$xv' + v = -\frac{1 + v^2}{2v},$$

which is separable and so leads to

$$\int \frac{1}{-(1+v^2)/(2v)-v}\,dv = \int \frac{1}{x}\,dx.$$

A little algebra then gives

$$-\int \frac{2v}{3v^2+1}\,dv = \int \frac{1}{x}\,dx.$$

Making the substitution $w = 3v^2+1$, the integral on the left-hand side transforms to give

$$-\int \frac{1/3}{w}\,dw = \int \frac{1}{x}\,dx,$$

and hence

$$-\tfrac{1}{3}\ln|w| = \ln|x| + c_1$$

for any constant c_1. From $w = 3v^2+1 = 3y^2/x^2+1$ comes the general implicit solution

$$\ln\left(\frac{3y^2}{x^2}+1\right) = -3\ln|x| + c_1,$$

which can be rearranged to give

$$3\ln|x| + \ln\left(\frac{3y^2}{x^2}+1\right) = \ln|x^3| + \ln\left(\frac{3y^2}{x^2}+1\right) = \ln\left(3|x|y^2+|x|x^2\right) = c_1.$$

To get the general solution in a nicer form, we can exponentiate the last equality to get

$$|x|(3y^2+x^2) = c_2,$$

where $c_2 = \exp(c_1) > 0$ is arbitrary. From this comes $x(3y^2+x^2) = \pm c_2$, and so we may replace $\pm c_2$ with the arbitrary constant $c \neq 0$ and write

$$x^3 + 3xy^2 = c.$$

11 Rewrite equation thus: $y' + y = xy^4$. This is Bernoulli with $n = 4$, $P(x) = 1$, and $Q(x) = x$. Letting $v = y^{1-n} = y^{-3}$, we obtain the linear equation

$$v' - 3v = -3x.$$

Multiplying by the integrating factor $\mu(x) = e^{-3x}$ yields

$$e^{-3x}v' - 3e^{-3x}v = -3xe^{-3x} \quad\Rightarrow\quad (e^{-3x}v)' = -3xe^{-3x} \quad\Rightarrow\quad e^{-3x}v = -3\int xe^{-3x}dx,$$

whence

$$e^{-3x}v = -3\left[-\tfrac{1}{3}xe^{-3x} - \tfrac{1}{9}e^{-3x} + c\right] = xe^{-3x} + \tfrac{1}{3}e^{-3x} + c \quad\Rightarrow\quad v(x) = x + \tfrac{1}{3} + ce^{3x},$$

and finally

$$y^{-3} = x + \frac{1}{3} + ce^{3x} \qquad\text{or}\qquad y = \sqrt[3]{\frac{3}{3x+1+ce^{3x}}}.$$

Also $y \equiv 0$ is a solution.