1 Substitute $\varphi(x) = e^{mx}$ for $y$ to obtain

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = 0 \Rightarrow 2m^2 + 7m - 4 = 0 \Rightarrow (2m - 1)(m + 4) = 0,$$

so $m \in \{1/2, -4\}$, and therefore $\varphi_1(x) = e^{x/2}$ and $\varphi_2(x) = e^{-4x}$ are solutions to the ODE. (Note that $y \equiv 0$ is a solution also.)

2 Given $y = \sin x,$

$$y' = \sqrt{1 - y^2} \iff \cos x = \sqrt{1 - \sin^2 x} \iff \cos x = \sqrt{\cos^2 x} \iff \cos x = |\cos x|.$$

Thus we must have $\cos x \geq 0,$ and since we generally take solutions to differential equations to be defined on open intervals we must have $x \in (-\pi/2 + 2\pi n, \pi/2 + 2\pi n)$ for any $n \in \mathbb{Z}.$ So we may let $I = (-\pi/2, \pi/2).$

3 Rewrite equation as $y' = x^2/(1 + y^3).$ According to the Existence-Uniqueness Theorem the IVP

$$y' = \frac{x^2}{1 + y^3}, \quad y(x_0) = y_0,$$

has a unique solution if

$$f(x, y) = \frac{x^2}{1 + y^3} \quad \text{and} \quad f_y(x, y) = -\frac{3x^2y^2}{(1 + y^3)^2}$$

are both continuous on some open set in $\mathbb{R}^2$ containing $(x_0, y_0).$ This will be the case for any $(x_0, y_0) \in \mathbb{R}^2$ with $y_0 \neq -1.$

4 Here $x'(t)$ denotes the rate of change of the drug’s amount in the bloodstream with respect to time $t,$ and so if we let $k$ be the constant of proportionality we have $x'(t) = r - kx(t).$

5 Rewriting the equation as

$$y' = -y^2e^{\cos x} \sin x$$

shows it to be separable. We obtain

$$-\int \frac{1}{y^2} dy = \int e^{\cos x} \sin x \, dx.$$

Let $u = \cos x$ in the integral at right, so

$$\frac{1}{y} = -\int e^u du = -e^u + c = -e^{\cos x} + c$$

for arbitrary constant $c.$ That is,

$$y = \frac{1}{c - e^{\cos x}}.$$
Writing the equation as \( y' = 2 \cos x \sqrt{y+1} \), we see the equation is separable. We get
\[
\int \frac{1}{\sqrt{y+1}} \, dy = \int 2 \cos x \, dx.
\]
This integrates easily to give
\[
2\sqrt{y+1} = 2 \sin x + c.
\]
Now, \( y(\pi) = 0 \) implies that
\[
2\sqrt{0+1} = 2 \sin \pi + c,
\]
or \( c = 2 \). The (implicit) solution to the IVP is thus
\[
2\sqrt{y+1} = 2 \sin x + 2,
\]
or \( y = (\sin x + 1)^2 - 1 \).

The equation may be written as
\[
y' + \frac{3}{x} y = \frac{\sin x}{x^2} - 3x,
\]
which is the standard form for a 1st-order linear ODE. An integrating factor is
\[
\mu(x) = e^{\int \frac{3}{x} \, dx} = e^{3 \ln x} = x^3.
\]
Multiplying the ODE by \( x^3 \) gives
\[
x^3 y' + 3x^2 y = x \sin x - 3x^4,
\]
which becomes \((x^3 y)' = x \sin x - 3x^4\) and thus
\[
x^3 y = \int x \sin x \, dx - \frac{3}{5} x^5 + c.
\]
Integration by parts gives
\[
\int x \sin x \, dx = \sin x - x \cos x,
\]
so that \( x^3 y = \sin x - x \cos x - \frac{3}{5} x^5 + c \) and therefore
\[
y(x) = \frac{1}{x^3} \left( \sin x - x \cos x - \frac{3}{5} x^5 + c \right).
\]
is the general solution.

The equation may be written as
\[
x' + \frac{3}{t} x = t^2 \ln t + \frac{1}{t^2},
\]
which is the standard form for a 1st-order linear ODE. An integrating factor is
\[
\mu(x) = e^{\int \frac{3}{t} \, dt} = t^3.
\]
Multiplying the ODE by \( t^3 \) gives \( t^3 x' + 3t^2 x = t^5 \ln t + t \), which becomes \((t^3 x)' = t^5 \ln t + t\) and thus
\[
t^3 x = \int t^5 \ln t \, dt + \frac{1}{2} t^2 + c.
\]
By integration by parts we find that
\[
\int t^5 \ln t \, dt = \frac{1}{6} t^6 \ln t - \int \frac{1}{6} t^5 \, dt = \frac{t^6}{6} \ln t - \frac{t^6}{36},
\]
and so we have
\[
t^3 x = \frac{t^6}{6} \ln t - \frac{t^6}{36} + \frac{1}{2} t^2 + c.
\]
Letting \( t = 1 \) and \( x = 0 \) (the initial condition) gives \( 0 = -\frac{1}{36} + \frac{1}{2} + c \), so that \( c = -\frac{17}{36} \) and we obtain
\[
x(t) = \frac{1}{6} t^3 \left( \ln t - \frac{1}{6} \right) + \frac{1}{2t} - \frac{17}{36t^3}
\]
as the solution to the IVP.

9 We have
\[
M(x, y) = x - y^3 + y^2 \sin x \quad \text{and} \quad N(x, y) = -3xy^2 - 2y \cos x.
\]
Since the equation is exact there exists a function \( F(x, y) \) such that \( F_x = M \) and \( F_y = N \); that is,
\[
F_x(x, y) = x - y^3 + y^2 \sin x \quad \text{and} \quad F_y(x, y) = -3xy^2 - 2y \cos x. \tag{1}
\]
Integrate the first equation in (1) with respect to \( x \):
\[
F(x, y) = \int (x - y^3 + y^2 \sin x) \, dx + g(y) = \frac{1}{2} x^2 - xy^3 - y^2 \cos x + g(y). \tag{2}
\]
Differentiating this with respect to \( y \) yields
\[
F_y(x, y) = -3xy^2 - 2y \cos x + g'(y),
\]
and so from the second equation in (1) we obtain
\[
-3xy^2 - 2y \cos x + g'(y) = -3xy^2 - 2y \cos x,
\]
or simply \( g'(y) = 0 \). Hence \( g(y) = c_1 \) for some arbitrary constant \( c_1 \), and so (2) becomes
\[
F(x, y) = \frac{1}{2} x^2 - xy^3 - y^2 \cos x + c_1.
\]
The general implicit solution to the ODE is therefore
\[
\frac{1}{2} x^2 - xy^3 - y^2 \cos x + c_1 = c_2
\]
for arbitrary \( c_2 \), which we can write simply as
\[
\frac{1}{2} x^2 - xy^3 - y^2 \cos x = c
\]
by consolidating the arbitrary constants \( c_1 \) and \( c_2 \).

10 Rewrite ODE as
\[
y' = \frac{x^2 + y^2}{2xy} = -\frac{1 + (y/x)^2}{2(y/x)}.
\]
Let \( v = y/x \), so \( y = xv \) and we get \( y' = xv' + v \). ODE then becomes
\[
xv' + v = -\frac{1 + v^2}{2v},
\]
which is separable and so leads to
\[ \int \frac{1}{-(1+v^2)/(2v)} \, dv = \int \frac{1}{x} \, dx. \]
A little algebra then gives
\[ -\int \frac{2v}{3v^2 + 1} \, dv = \int \frac{1}{x} \, dx. \]
Making the substitution \( w = 3v^2 + 1 \), the integral on the left-hand side transforms to give
\[ -\int \frac{1}{3} \, dw = \int \frac{1}{x} \, dx, \]
and hence
\[ -\frac{1}{3} \ln |w| = \ln |x| + c_1 \]
for any constant \( c_1 \). From \( w = 3v^2 + 1 = 3y^2/x^2 + 1 \) comes the general implicit solution
\[ \ln \left( \frac{3y^2}{x^2} + 1 \right) = -3 \ln |x| + c_1, \]
which can be rearranged to give
\[ 3 \ln |x| + \ln \left( \frac{3y^2}{x^2} + 1 \right) = \ln |x^3| + \ln \left( \frac{3y^2}{x^2} + 1 \right) = \ln \left( 3|x|y^2 + |x|x^2 \right) = c_1. \]

To get the general solution in a nicer form, we can exponentiate the last equality to get
\[ |x|(3y^2 + x^2) = c_2, \]
where \( c_2 = \exp(c_1) > 0 \) is arbitrary. From this comes \( x(3y^2 + x^2) = \pm c_2 \), and so we may replace \( \pm c_2 \) with the arbitrary constant \( c \neq 0 \) and write
\[ x^3 + 3xy^2 = c. \]

11 Rewrite equation thus: \( y' + y = xy^4 \). This is Bernoulli with \( n = 4 \), \( P(x) = 1 \), and \( Q(x) = x \). Letting \( v = y^{1-n} = y^{-3} \), we obtain the linear equation
\[ v' - 3v = -3x. \]
Multiplying by the integrating factor \( \mu(x) = e^{-3x} \) yields
\[ e^{-3x}v' - 3e^{-3x}v = -3xe^{-3x} \Rightarrow (e^{-3x}v)' = -3xe^{-3x} \Rightarrow e^{-3x}v = -3 \int xe^{-3x} \, dx, \]
whence
\[ e^{-3x}v = -3 \left[ -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c \right] = xe^{-3x} + \frac{1}{3}e^{-3x} + c \Rightarrow v(x) = x + \frac{1}{3} + ce^{3x}, \]
and finally
\[ y^{-3} = x + \frac{1}{3} + ce^{3x} \quad \text{or} \quad y = \sqrt[3]{\frac{3}{3x + 1 + ce^{3x}}}. \]
Also \( y \equiv 0 \) is a solution.