Math 250 Exam #2 Key (Summer 2014)

1 Rewrite ODE as

$$y' = \frac{y}{x} - \left(\frac{y}{x}\right)^{-2},$$

let $v = y/x$, so $y = xv$ and we get $y' = xv' + v$. ODE then becomes

$$xv' + v = v - \frac{1}{v^2},$$

and thus $xv' = -v^{-2}$. The equation is separable, leading to

$$-\int v^2 dv = \int \frac{1}{x} dx,$$

and then

$$-\frac{1}{3}v^3 = \ln|x| + c \quad \Rightarrow \quad v^3 = c - \ln|x^3| \quad \Rightarrow \quad \frac{y^3}{x^3} = c - 3\ln|x|.$$

The solution curve must contain the point $(1, 2)$, and since $x > 0$ at this point, we have $|x| = x$. Thus $y^3/x^3 = c - 3\ln x$, and from the initial condition we find that $c = 8$. Therefore

$$y^3 = x^3(8 - 3\ln x)$$

is the solution.

2 The equation is Bernoulli with $n = 4$, $P(x) = 1$, and $Q(x) = x$. Letting $v = y^{1-n} = y^{-3}$, we obtain the linear equation

$$v' - 3v = -3x.$$

Multiplying by the integrating factor

$$\mu(x) = e^{-\int 3\,dx} = e^{-3x}$$

yields

$$v'e^{-3x} - 3ve^{-3x} = -3xe^{-3x} \quad \Rightarrow \quad (ve^{-3x})' = -3xe^{-3x} \quad \Rightarrow \quad ve^{-3x} = -3\int xe^{-3x}dx,$$

and thus

$$\frac{v}{e^{3x}} = \frac{3x+1}{3e^{3x}} + c.$$

Therefore

$$y^{-3} = ce^{3x} + x + \frac{1}{3}.$$

3 Let $x(t)$ be the mass of salt, in kilograms, in the tank at time t, so that $x(0) = 30$. The volume of solution in the tank is $V(t) = 200 + 2t$. The full derivation of $x'(t)$, which is the rate of change of the amount of salt in the tank at time t, is as follows:

$$\begin{aligned} x'(t) &= (\text{rate salt enters Tank 1}) - (\text{rate salt leaves Tank 1}) \\ &= \left(\frac{0.3\text{ kg}}{1\text{ L}}\right)\left(\frac{4\text{ L}}{1\text{ min}}\right) - \left(\frac{x(t)\text{ kg}}{V(t)\text{ L}}\right)\left(\frac{2\text{ L}}{1\text{ min}}\right) \end{aligned}$$

$$= 1.2 - \frac{2x(t)}{200 + 2t}.$$

Thus we have a linear first-order ODE:

$$x' + \frac{x}{t + 100} = \frac{6}{5}.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{1}{t + 100}\, dt\right) = e^{\ln(t+100)} = t + 100$$

to obtain

$$(t + 100)x' + x = \frac{6}{5}(t + 100),$$

which becomes

$$[(t + 100)x]' = \frac{6}{5}(t + 100)$$

and thus

$$(t + 100)x = \frac{6}{5}\int (t + 100)\, dt = \frac{3}{5}t^2 + 120t + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{3t^2 + 600t + c}{5t + 500}.$$

To determine c we use the initial condition $x(0) = 30$, giving $c/500 = 30$, and thus $c = 15,000$. So, the amount of salt in the tank at time t is given by

$$x(t) = \frac{3t^2 + 600t + 15,000}{5t + 500}.$$

The tank is full when $t = 150$ minutes. At that time the concentration of salt is:

$$\frac{x(150)}{V(150)} = \frac{1}{500}\left(\frac{3(150)^2 + 600(150) + 15,000}{5(150) + 500}\right) = \frac{138}{500} = 0.276 \text{ kg/L}.$$

4 Auxiliary equation is $2r^2 + 7r - 15 = 0$, which has solutions $r = \frac{3}{2}, -5$. Hence

$$y(t) = c_1 e^{3t/2} + c_2 e^{-5t}.$$

Now,

$$y'(t) = \tfrac{3}{2}c_1 e^{3t/2} - 5c_2 e^{-5t},$$

and so the initial conditions give

$$c_1 + c_2 = -2 \quad \text{and} \quad \tfrac{3}{2}c_1 - 5c_2 = 4.$$

Solving the system gives $c_1 = -\frac{12}{13}$ and $c_2 = -\frac{14}{13}$, and therefore

$$y(t) = -\tfrac{12}{13}e^{3t/2} - \tfrac{14}{13}e^{-5t}.$$

5 Auxiliary equation is $9r^2 - 12r + 4 = 0$, which has double root $r = 2/3$. General solution is therefore

$$y(t) = c_1 e^{2t/3} + c_2 t e^{2t/3}.$$

6 Auxiliary equation is $12r^3 - 28r^2 - 3r + 7 = 0$, which factors by grouping:

$$(3r - 7)(2r - 1)(2r + 1) = 0.$$

Roots are $r = 7/3, 1/2, -1/2$. General solution:

$$y(t) = c_1 e^{-t/2} + c_2 e^{t/2} + c_3 e^{7t/3}.$$

7 Auxiliary equation is $r^2 + 9 = 0$, which has roots $r = \pm 3i$. Thus we have

$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$

With the initial conditions $y(0) = 1$ and $y'(0) = 1$ we find that $c_1 = 1$ and $c_2 = 1/3$, and so

$$y(t) = \cos 3t + \tfrac{1}{3} \sin 3t.$$

8a We have $y'' + y' + 4y = e^t + e^{-t}$.

$$\text{Auxiliary equation: } r^2 + r + 4 = 0; \quad \text{roots: } -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i.$$

Start with equation $y'' + y' + 4y = e^t$, with nonhomogeneity $f(t) = e^t$. Since $\alpha = 1$ is not a root of the auxiliary equation, a particular solution has form $y_1(t) = Ae^t$. Substituting into ODE:

$$Ae^t + (Ae^t)' + 4(Ae^t)'' = e^t \quad \Rightarrow \quad 6Ae^t = e^t \quad \Rightarrow \quad A = \frac{1}{6},$$

so $y_1(t) = \frac{1}{6} e^t$.

Next we have $y'' + y' + 4y = e^{-t}$, with nonhomogeneity $f(t) = e^{-t}$. Since $\alpha = -1$ is not a root of the auxiliary equation, a particular solution has form $y_2(t) = Ae^{-t}$. Substituting into ODE:

$$Ae^{-t} + (Ae^{-t})' + 4(Ae^{-t})'' = e^{-t} \quad \Rightarrow \quad 4Ae^{-t} = e^{-t} \quad \Rightarrow \quad A = \frac{1}{4},$$

so $y_2(t) = \frac{1}{4} e^{-t}$.

By Superposition Principle a particular solution to the original equation is

$$y_p(t) = \frac{1}{6} e^t + \frac{1}{4} e^{-t}.$$

The general solution is thus

$$y(t) = \frac{1}{6} e^t + \frac{1}{4} e^{-t} + e^{-t/2} \left(c_1 \cos \frac{\sqrt{15}}{2} t + c_2 \sin \frac{\sqrt{15}}{2} t \right).$$

8b The auxiliary equation $r^2 + 2r + 5 = 0$ has roots $-1 \pm 2i$. The nonhomogeneity has the factor $e^{\alpha t} \cos \beta t$ with $\alpha - 1$ and $\beta = 2$, and since $\alpha + \beta i = -1 + 2i$ is a root of the auxiliary equation, we take $s = 1$ in the Method of Undetermined Coefficients. Thus a particular solution has form

$$y_p(t) = Ate^{-t} \cos 2t + Bte^{-t} \sin 2t.$$

Now $y_p''(t) + 2y_p'(t) + 5y_p(t) = 4e^{-t} \cos 2t$ becomes

$$(-4A \sin 2t + 4B \cos 2t)e^{-t} = (4 \cos 2t)e^{-t},$$

which shows that $A = 0$ and $B = 1$, and therefore

$$y_p(t) = te^{-t} \sin 2t.$$

The general solution is thus

$$y(t) = te^{-t} \sin 2t + e^{-t}(c_1 \cos 2t + c_2 \sin 2t). \tag{1}$$

Next, substituting the initial condition $y(0) = 1$ into (1) yields $c_1 = 1$. Substituting the initial condition $y'(0) = 0$ into the derivative of (1) yields $c_2 = \frac{1}{2}$. We now obtain the solution to the IVP:

$$y(t) = te^{-t} \sin 2t + e^{-t}\left(\cos 2t + \tfrac{1}{2} \sin 2t\right)$$

9 The auxiliary equation $r^2 - 2r + 1 = 0$ has double root 1, so the corresponding homogeneous equation $y'' - 2y' + 1 = 0$ has linearly independent solutions $y_1(t) = e^t$ and $y_2(t) = te^t$, and thus

$$y_1'(t) = e^t \quad \text{and} \quad y_2'(t) = (1 + t)e^t.$$

Now, since $a_2 = 1$ and $f(t) = (1 + t^2)^{-1} e^t$, we have

$$v_1(t) = \int \frac{-te^t \cdot (1 + t^2)^{-1} e^t}{e^{2t}} dt = -\int \frac{t}{1 + t^2} dt = -\frac{1}{2} \ln(1 + t^2),$$

and

$$v_2(t) = \int \frac{e^t \cdot (1 + t^2)^{-1} e^t}{e^{2t}} dt = \int \frac{1}{1 + t^2} dt = \tan^{-1}(t).$$

Hence a particular solution to the ODE is

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = -\tfrac{1}{2} e^t \ln(1 + t^2) + te^t \tan^{-1}(t).$$

General solution is therefore

$$y(t) = \left[-\tfrac{1}{2} \ln(1 + t^2) + t \tan^{-1}(t) + c_1 t + c_2\right] e^t.$$