Math 250 Exam #1 Key (Summer 2014)

1 Substitute $\varphi(x) = x^m$ for y to obtain

$$\begin{aligned} 3x^2 \cdot m(m-1)x^{m-2} + 11x \cdot mx^{m-1} - 3x^m &= 0 \\ (3m^2 - 3m)x^m + 11mx^m - 3x^m &= 0 \\ (3m^2 + 8m - 3)x^m &= 0 \end{aligned}$$

To satisfy the equation for all x in some interval $I \subseteq \mathbb{R}$, it will be necessary to have

$$3m^2 + 8m - 3 = 0.$$

Solving this equation for m, we have

$$(3m-1)(m+3) = 0$$

and thus $m = 1/3, -3$. This shows that $\varphi_1(x) = \sqrt[3]{x}$ and $\varphi_2(x) = x^{-3}$ are solutions to the ODE (each valid on $(-\infty, 0) \cup (0, \infty)$, incidentally).

2 We have

$$f(x,y) = 4x^2 + \sqrt[3]{2-y},$$

which is continuous throughout $\mathbb{R}^2$. However

$$f_y(x,y) = \frac{1}{3}(2-y)^{-2/3} = \frac{1}{3\sqrt[3]{(2-y)^2}}$$

is not continuous on the line $y = 2$ since f_y is not defined there. The initial point $(-1, 2)$ lies on this line, and so the Existence-Uniqueness Theorem does not imply that the initial value problem has a unique solution.

3 We rewrite the equation as

$$ye^y \frac{dy}{dx} = \frac{1 + e^{-2x}}{e^x},$$

which is separable:

$$\int ye^y \, dy = \int (e^{-x} + e^{-3x}) dx.$$

Integrating gives

$$ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + c.$$

4 Again the equation is separable:

$$\int \frac{1}{x^2+1} dx = \int 4 \, dt \quad \Rightarrow \quad \tan^{-1}(x) = 4t + c.$$

From the initial condition $x(\pi/4) = 1$ we obtain $\tan^{-1}(1) = \pi + c$, and thus $c = -3\pi/4$. Solution is

$$\tan^{-1}(x) = 4t - \frac{3\pi}{4}.$$

5 The equation may be written as

$$y' + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x,$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3.$$

Multiplying the ODE by x^3 gives

$$x^3y' + 3x^2y = x\sin x - 3x^4,$$

which becomes $(x^3y)' = x\sin x - 3x^4$ and thus

$$x^3y = \int x\sin x\,dx - \frac{3}{5}x^5 + c.$$

Integration by parts gives

$$\int x\sin x\,dx = \sin x - x\cos x,$$

so that $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + c$ and therefore

$$y(x) = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + c\right).$$

is the general solution.

6 The equation may be written as

$$x' + \frac{3}{t}x = t^2\ln t + \frac{1}{t^2},$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/t\,dt} = t^3.$$

Multiplying the ODE by t^3 gives $t^3x' + 3t^2x = t^5\ln t + t$, which becomes $(t^3x)' = t^5\ln t + t$ and thus

$$t^3x = \int t^5\ln t\,dt + \frac{1}{2}t^2 + c.$$

By integration by parts we find that

$$\int t^5\ln t\,dt = \frac{1}{6}t^6\ln t - \int\frac{1}{6}t^5\,dt = \frac{t^6}{6}\ln t - \frac{t^6}{36},$$

and so we have

$$t^3x = \frac{t^6}{6}\ln t - \frac{t^6}{36} + \frac{1}{2}t^2 + c.$$

Letting $t = 1$ and $x = 0$ (the initial condition) gives $0 = -\frac{1}{36} + \frac{1}{2} + c$, so that $c = -\frac{17}{36}$ and we obtain

$$x(t) = \frac{1}{6}t^3\left(\ln t - \frac{1}{6}\right) + \frac{1}{2t} - \frac{17}{36t^3}$$

as the solution to the IVP.

7 We have

$$M(x,y) = 1 + \ln y \quad \text{and} \quad N(x,y) = \frac{x}{y}.$$

Since the equation is exact there exists a function $F(x,y)$ such that $F_x = M$ and $F_y = N$; that is,

$$F_x(x,y) = 1 + \ln y \quad \text{and} \quad F_y(x,y) = \frac{x}{y}. \tag{1}$$

Integrate the first equation in (1) with respect to x:

$$F(x,y) = \int (1 + \ln y)dx + g(y) = x(1 + \ln y) + g(y). \tag{2}$$

Differentiating this with respect to y yields

$$F_y(x,y) = \frac{x}{y} + g'(y),$$

and so using the second equation in (1) we obtain

$$\frac{x}{y} + g'(y) = \frac{x}{y},$$

or simply $g'(y) = 0$. Hence $g(y) = c_1$ for some arbitrary constant c_1, and so (2) becomes

$$F(x,y) = x(1 + \ln y) + c_1.$$

The general implicit solution to the ODE is therefore

$$x(1 + \ln y) + c_1 = c_2$$

for arbitrary c_2, which we can write simply as

$$x + x \ln y = c$$

by consolidating the arbitrary constants c_1 and c_2.

8 We have

$$\frac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \frac{2y - 1}{y} = 2 - \frac{1}{y}$$

is a function of only y, so

$$\mu(y) = \exp\left(\int (2 - 1/y)\, dy\right) = e^{2y - \ln y} = y^{-1}e^{2y}.$$

Multiplying the ODE by $y^{-1}e^{2y}$ yields the exact equation

$$e^{2y} + (2xe^{2y} - y^{-1})y' = 0.$$

There exists a function F such that

$$F_x(x,y) = e^{2y} \quad \text{and} \quad F_y(x,y) = 2xe^{2y} - y^{-1}.$$

From the former equation comes

$$F(x,y) = xe^{2y} + g(y),$$

so the latter equation implies

$$2xe^{2y} + g'(y) = 2xe^{2y} - y^{-1} \quad \Rightarrow \quad g'(y) = -y^{-1} \quad \Rightarrow \quad g(y) = -\ln|y| + c_1,$$

c_1 arbitrary. The general solution to the ODE is $F(x,y) = c_2$, where c_2 is arbitrary. That is,

$$xe^{2y} - \ln|y| = c,$$

where c is the arbitrary constant deriving from $c_2 - c_1$.

9 Multiply the ODE by $x^m y^n$:

$$(x^{m+3}y^{n+2} - 2x^m y^{n+3}) + (x^{m+4}y^{n+1})y' = 0.$$

For exactness we need $M_y = N_x$, or

$$(n+2)x^{m+3}y^{n+1} - 2(n+3)x^m y^{n+2} = (m+4)x^{m+3}y^{n+1}.$$

Matching coefficients of like terms, we find that we must have $n+2 = m+4$ and $-2(n+3) = 0$, which solves to give $m = -5$ and $n = -3$. Thus an integrating factor is $\mu(x,y) = x^{-5}y^{-3}$, and the ODE becomes

$$(x^{-2}y^{-1} - 2x^{-5}) + (x^{-1}y^{-2})y' = 0,$$

which is exact. We now find a function F such that $F_x(x,y) = x^{-2}y^{-1} - 2x^{-5}$ and $F_y(x,y) = x^{-1}y^{-2}$. From the former equation we obtain

$$F(x,y) = \int \left(x^{-2}y^{-1} - 2x^{-5}\right) dx = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + g(y),$$

and from the latter equation comes

$$x^{-1}y^{-2} = F_y(x,y) = x^{-1}y^{-2} + g'(y),$$

or $g'(y) = 0$. Thus $g(y) = c_1$ for some constant c_1, and we have

$$F(x,y) = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + c_1.$$

The implicit solution to the ODE is therefore $F(x,y) = c_2$ for arbitrary constant c_2, which we may write simply as

$$\tfrac{1}{2}x^{-4} - x^{-1}y^{-1} = c$$

by merging constant terms. Another solution is $y \equiv 0$.