Math 250 Exam #3 Key (Summer 2013)

1 The model for the mass-spring system is

$$20y'' + 140y' + 200y = 0, \quad y(0) = 0.25, \quad y'(0) = -1.$$

The auxiliary equation is

$$20r^2 + 140r + 200 = 0,$$

or $r^2 + 7r + 10 = 0$, which has roots $-2, -5$. Thus the general solution to the ODE is $y(t) = c_1 e^{-2t} + c_2 e^{-5t}$. From the initial condition $y(0) = 0.25$ comes $c_1 + c_2 = 0.25$, and from $y'(t) = -2c_1 e^{-2t} - 5c_2 e^{-5t}$ and the initial condition $y'(0) = -1$ comes $-2c_1 - 5c_2 = -1$. Solving the system

$$\begin{cases} c_1 + c_2 = 0.25 \\ -2c_1 - 5c_2 = -1 \end{cases}$$

yields $c_1 = \frac{1}{12}$ and $c_2 = \frac{1}{6}$. Therefore

$$y(t) = \frac{1}{12}e^{-2t} + \frac{1}{6}e^{-5t}.$$

Since the equation $\frac{1}{12}e^{-2t} + \frac{1}{6}e^{-5t} = 0$ has no real solution, it follows that the object never returns to the equilibrium position.

2 There are two external forces acting on the object O: $F(t)$ and also gravity. Thus the total external force on O at time t is

$$F_{\text{ext}} = mg + F(t) = (8\text{ kg})(9.8\text{ m/s}^2) + \cos 2t\text{ N} = 78.4 + \cos 2t\text{ N}.$$

We need to determine the spring constant k. Upon attaching O to the spring, the spring stretches until its tension comes to equal the magnitude of the gravitational force acting on O. We have, by Hooke's Law, with the understanding that *down* is the *positive* direction,

$$-(8\text{ kg})(9.8\text{ m/s}^2) = -mg = -F_{\text{gravity}} = F_{\text{spring}} = -ky = -k(1.96\text{ m}),$$

and so $-1.96k = -78.4$, which yields a spring constant of $k = 40$ N/m. The model for the mass-spring system, $my'' + by' + ky = F_{\text{ext}}$, is thus

$$8y'' + 3y' + 40y = 78.4 + \cos 2t. \tag{1}$$

The form of a particular solution to

$$8y'' + 3y' + 40y = 78.4$$

is $y_{p_1}(t) = A$, where A is some constant. Now, $y'_{p_1}(t) = y''_{p_1}(t) = 0$, and so substitution into $8y'' + 3y' + 40y = 78.4$ gives $40A = 78.4$ and finally $A = 1.96$. Therefore $y_{p_1}(t) = 1.96$.

The form of a particular solution to

$$8y'' + 3y' + 40y = \cos 2t$$

is $y_{p_2}(t) = A\cos 2t + B\sin 2t$. Substituting this for y in the ODE yields

$$8(A\cos 2t + B\sin 2t)'' + 3(A\cos 2t + B\sin 2t)' + 40(A\cos 2t + B\sin 2t) = \cos 2t,$$

which gives

$$(8A + 6B)\cos 2t + (-6A + 8B)\sin 2t = \cos 2t,$$

and so we must have

$$\begin{cases} 8A + 6B = 1 \\ -6A + 8B = 0 \end{cases}$$

Solving this system of equations yields $A = 2/25$ and $B = 3/50$, and therefore

$$y_{p_2}(t) = \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t.$$

By the Superposition Principle a particular solution to (1) is $y_p = y_{p_1} + y_{p_2}$, or

$$y_p(t) = 1.96 + \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t, \tag{2}$$

which happens to be the steady-state solution for the mass-spring system. However, it is common practice to take the equilibrium position to be wherever the spring comes to rest once the mass is attached to it. We're given that the spring stretches 1.96 m, so we shift the point where y equals zero down by 1.96 m by subtracting 1.96 from the right-hand side of (2), resulting in

$$y_p(t) = \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t$$

as the steady-state solution. Alternatively we may write

$$y_p(t) = 0.1\sin(2t + \varphi),$$

where $\varphi = \arctan(4/3) \approx 0.927$.

3 We have

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st}\sin t\,dt.$$

Use integration by parts: $\int_a^b uv' = uv\big|_a^b - \int_a^b u'v$ with $u(x) = e^{-st}$ and $v'(x) = \sin t$ to get

$$\int_0^\pi e^{-st}\sin t\,dt = \left[-e^{-st}\cos t\right]_0^\pi - \int_0^\pi se^{-st}\cos t\,dt = 1 + e^{-\pi s} - s\int_0^\pi e^{-st}\cos t\,dt.$$

Use integration by parts on the rightmost integral with $u(x) = e^{-st}$ and $v'(x) = \cos t$ to get

$$\begin{aligned}\int_0^\pi e^{-st}\sin t\,dt &= 1 + e^{-\pi s} - s\left(\left[e^{-st}\sin t\right]_0^\pi - \int_0^\pi -se^{-st}\sin t\,dt\right) \\ &= 1 + e^{-\pi s} - s^2\int_0^\pi e^{-st}\sin t\,dt.\end{aligned}$$

Now isolate the integral:

$$(s^2+1)\int_0^\pi e^{-st}\sin t\,dt = 1 + e^{-\pi s} \quad\Rightarrow\quad \int_0^\pi e^{-st}\sin t\,dt = \frac{1+e^{-\pi s}}{s^2+1}$$

Therefore

$$\mathcal{L}[f](s) = \frac{1+e^{-\pi s}}{s^2+1}.$$

4 Using the table provided,

$$\mathcal{L}\left[t^2 e^{5t}\right](s) = \frac{2!}{(s-5)^{2+1}} = \frac{2}{(s-5)^3}$$

5 Using the table provided,

$$\begin{aligned}\mathcal{L}\left[t^5 - 7e^{-3t}\sin 4t\right](s) &= \mathcal{L}\left[t^5\right](s) - 7\mathcal{L}\left[e^{-3t}\sin 4t\right](s)\\ &= \frac{5!}{(s-0)^{5+1}} - 7\cdot\frac{4}{(s+3)^2+4^2}\\ &= \frac{120}{s^6} - \frac{28}{(s+3)^2+16}\end{aligned}$$

6 Use a trigonometric identity for this, along with the table provided:

$$\begin{aligned}\mathcal{L}\left[e^{8t}\cos^2 t\right](s) &= \mathcal{L}\left[e^{8t}\cdot\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2t\right)\right](s) = \tfrac{1}{2}\mathcal{L}\left[e^{8t}\right](s) + \tfrac{1}{2}\mathcal{L}\left[e^{8t}\cos 2t\right](s)\\ &= \frac{1}{2}\cdot\frac{0!}{(s-8)^{0+1}} + \frac{1}{2}\cdot\frac{s-8}{(s-8)^2+2^2} = \frac{1}{2(s-8)} + \frac{s-8}{2(s-8)^2+8}\end{aligned}$$

7 Partial fraction decomposition is necessary: we have

$$\frac{s+11}{(s-1)(s+3)} = \frac{3}{s-1} - \frac{2}{s+3},$$

and so

$$\mathcal{L}^{-1}\left[\frac{s+11}{(s-1)(s+3)}\right](t) = 3\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t) = 3e^t - 2e^{-3t}.$$

8 We have

$$\mathcal{L}[y''](s) + 2\mathcal{L}[y'](s) + 2\mathcal{L}[y](s) = \mathcal{L}\left[t^2\right](s) + 4\mathcal{L}[t](s).$$

Letting $Y(s) = \mathcal{L}[y(t)](s)$ and using relevant properties gives

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = \frac{2}{s^3} + \frac{4}{s^2}.$$

The initial conditions come next:

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}.$$

Solving for Y yields

$$Y = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}.$$

Now we employ partial fraction decomposition to obtain

$$Y = \left(\frac{1}{2s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{s/2}{s^2+2s+2}\right) + \left(-\frac{2}{s} + \frac{2}{s^2} + \frac{2s+2}{s^2+2s+2}\right) - \frac{1}{s^2+2s+2}$$

$$= -\frac{3/2}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{3s/2+1}{(s+1)^2+1^2}$$
$$= -\frac{3/2}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{3s/2+3/2}{(s+1)^2+1} - \frac{1/2}{(s+1)^2+1}.$$

Finally, we take the inverse Laplace transform of each side to get

$$y(t) = -\frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^3}\right](t) + \frac{3}{2}\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right](t)$$
$$-\frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right](t)$$
$$= -\frac{3}{2} + t + \frac{1}{2}t^2 + \frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t.$$

9 Taking the Laplace transform of both sides of the ODE yields

$$\left[s^3\mathcal{L}[y(t)](s) - s^2y(0) - sy'(0) - y''(0)\right] - \left[s^2\mathcal{L}[y(t)](s) - sy(0) - y'(0)\right]$$
$$+ \left[s\mathcal{L}[y(t)](s) - y(0)\right] - \mathcal{L}[y(t)](s) = \mathcal{L}[0](s).$$

Letting $Y(s) = \mathcal{L}[y(t)](s)$ and noting that $\mathcal{L}[0](s) = 0$, we use the initial conditions to obtain

$$[s^3Y(s) - s^2 - s - 3] - [s^2Y(s) - s - 1] + [sY(s) - 1] - Y(s) = 0,$$

and thus

$$Y(s) = \frac{s^2+3}{s^3 - s^2 + s - 1} = \frac{s^2+3}{(s-1)(s^2+1)}.$$

The partial fraction decomposition of the rational expression on the right-hand has the form

$$\frac{s^2+3}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1},$$

whence

$$s^2 + 3 = A(s^2+1) + (Bs+C)(s-1) = (A+B)s^2 + (C-B)s + (A-C).$$

This gives rise to the system

$$\begin{cases} A + B \phantom{{}+C} = 1 \\ - B + C = 0 \\ A \phantom{{}-B} - C = 3 \end{cases}$$

which has solution $(A, B, C) = (2, -1, -1)$, and so

$$Y(s) = \frac{2}{s-1} - \frac{s+1}{s^2+1} = \frac{2}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}.$$

Finally,

$$y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t)$$

leads to

$$y(t) = 2e^t - \cos t - \sin t$$

as the solution to the IVP.