Math 250 Exam #2 Key (Summer 2013)

1 Solve for y':

$$y' = -\frac{x^2+y^2}{2xy} = -\frac{x}{2y} - \frac{y}{2x}. \tag{1}$$

Let $v = y/x$, so that $y = vx$. Differentiating with respect to x yields $y' = (vx)' = v'x + v$, and so (1) becomes

$$v'x + v = -\frac{1}{2v} - \frac{v}{2} \quad\Rightarrow\quad xv' = -\frac{3v^2+1}{2v} \quad\Rightarrow\quad v' = \frac{1}{x}\left(-\frac{3v^2+1}{2v}\right)$$

This is a separable equation. We obtain

$$\int -\frac{2v}{3v^2+1}dv = \int \frac{1}{x}dx.$$

Performing the substitution $u = 3v^2+1$ on the left-hand integral gives

$$-\int \frac{1/3}{u}du = \int \frac{1}{x}dx \quad\Rightarrow\quad -\frac{1}{3}\ln|u| = \ln|x| + c \quad\Rightarrow\quad -\frac{1}{3}\ln(3v^2+1) = \ln|x| + c,$$

and hence

$$-\frac{1}{3}\ln\left(\frac{3y^2}{x^2}+1\right) = \ln|x| + c \quad\Rightarrow\quad \ln|x| + \frac{1}{3}\ln\left(\frac{3y^2}{x^2}+1\right) = c \quad\Rightarrow\quad \ln\left(|x|\sqrt[3]{\frac{3y^2}{x^2}+1}\right) = c$$

for arbitrary $c \in \mathbb{R}$. Exponentiating both sides yields

$$|x|\sqrt[3]{\frac{3y^2}{x^2}+1} = \tilde{c},$$

where $\tilde{c} = e^c > 0$ is arbitrary. Thus

$$x\sqrt[3]{\frac{3y^2}{x^2}+1} = \pm\tilde{c} = \hat{c},$$

where $\hat{c} \neq 0$ is arbitrary. Cubing both sides yields

$$x^3\left(\frac{3y^2}{x^2}+1\right) = \hat{c} \quad\Rightarrow\quad 3xy^2 + x^3 = \hat{c}.$$

2 In the standard form $y' + P(x)y = Q(x)y^n$ we have

$$y' - \frac{2}{x}y = -x^2y^2,$$

so $P(x) = -2/x$, $Q(x) = -x^2$, and $n = 2$. Letting $v = y^{1-n} = y^{-1}$, the equation will transform into $v' + (1-n)P(x)v = (1-n)Q(x)$:

$$v' + \frac{2}{x}v = x^2. \tag{2}$$

This is a linear equation. A suitable integrating factor is

$$\mu(x) = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln|x|} = e^{\ln x^2} = x^2.$$

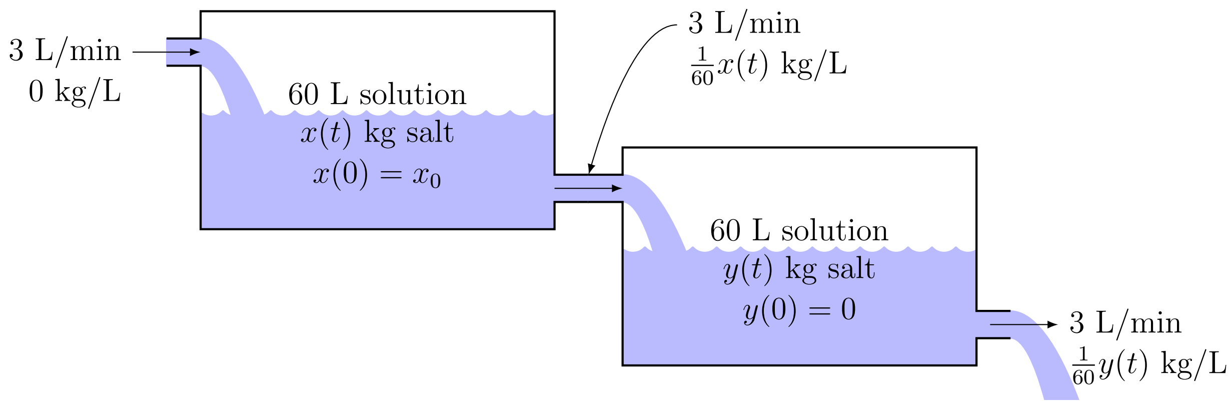
Multiply (2) by x^2 to get

$$x^2v' + 2xv = x^4 \quad\Rightarrow\quad (x^2v)' = x^4 \quad\Rightarrow\quad x^2v = \int x^4 dx \quad\Rightarrow\quad x^2v = \frac{1}{5}x^5 + c,$$

whence

$$\frac{x^2}{y} = \frac{x^5}{5} + c.$$

3 Let $x(t)$ be the number of kg of salt in Tank 1 at time t, and let $y(t)$ be the number of kg of salt in Tank 2 at time t. We have $x(0) = x_0$ for some constant x_0, and also $y(0) = 0$ since the water in Tank 2 is initially pure.



The volume of solution in Tank 1 is a constant 60 L, so

$$\begin{aligned} x'(t) &= (\text{rate salt enters Tank 1}) - (\text{rate salt leaves Tank 1}) \\ &= 0 - \left(\frac{x(t)\ \text{kg}}{60\ \text{L}}\right)\left(\frac{3\ \text{L}}{1\ \text{min}}\right) = -\frac{3x(t)}{60}, \end{aligned}$$

which yields the equation $x' = -\frac{1}{20}x$, also written as $dx/dt = -x/20$. This equation is separable, giving

$$\int \frac{20}{x}\,dx = -\int dt,$$

and hence

$$\ln\left(x^{20}\right) = -t + c_0$$

for arbitrary constant c_0. Exponentiating both sides and letting $c_1 = e^{c_0}$ be an arbitrary positive constant, we obtain

$$x^{20} = e^{-t+c_0} = c_1e^{-t},$$

and then $x(t) = c_1e^{-t/20}$. Using the initial condition $x(0) = x_0$, we substitute $t = 0$ and $x = x_0$ into the equation to get $x_0 = c_1e^0 = c_1$, and thus

$$x(t) = x_0e^{-t/20}. \tag{3}$$

Now we turn our attention to Tank 2. Since the volume of solution in Tank 2 is always 60 L, we have

$$y'(t) = (\text{rate salt enters Tank 2}) - (\text{rate salt leaves Tank 2})$$

$$= \left(\frac{x(t)\text{ kg}}{60\text{ L}}\right)\left(\frac{3\text{ L}}{1\text{ min}}\right) - \left(\frac{y(t)\text{ kg}}{60\text{ L}}\right)\left(\frac{3\text{ L}}{1\text{ min}}\right)$$

$$= \frac{x(t)}{20} - \frac{y(t)}{20} = \frac{x_0 e^{-t/20} - y(t)}{20}$$

where the last equality follows from (3). Hence we have the equation

$$y' + \frac{1}{20}y = \frac{x_0}{20}e^{-t/20}, \tag{4}$$

which is a first-order linear ODE and so can be solved by finding an appropriate integrating factor $\mu(t)$. We have

$$\mu(t) = \exp\left(\int \frac{1}{20}\,dt\right) = e^{t/20}$$

and so, multiplying (4) by $e^{t/20}$, we obtain

$$y'e^{t/20} + \frac{1}{20}ye^{t/20} = \frac{x_0}{20},$$

which can be written

$$\left(ye^{t/20}\right)' = \frac{x_0}{20},$$

and therefore

$$ye^{t/20} = \int \frac{x_0}{20}\,dt = \frac{x_0}{20}t + c.$$

Using the initial condition $y(0) = 0$, we substitute $t = 0$ and $y = 0$ into this equation to find that $c = 0$, and at last we have an expression for $y(t)$:

$$y(t) = \frac{x_0}{20}te^{-t/20}.$$

The *concentration* of salt in Tank 2 at time t, $C(t)$, is given by $C(t) = y(t)/60$; that is,

$$C(t) = \frac{x_0}{1200}te^{-t/20}.$$

To determine when the concentration is greatest, we must find $t > 0$ for which $C(t)$ attains a global maximum value on $(0, \infty)$. This entails finding t for which $C'(t) = 0$; that is, we must solve

$$\frac{x_0}{1200}e^{-t/20} - \frac{x_0}{24,000}te^{-t/20} = 0.$$

But this equation immediately implies that $20 - t = 0$, and hence $t = 20$ minutes.

4 Suppose that $c_1\varphi + c_2\psi + c_3\xi = 0$ on $(-\infty, \infty)$. This means that $c_1te^{2t} + c_2e^{2t} + c_3e^t = 0$ for all $t \in (-\infty, \infty)$. If we let t equal 0, 1, and 2, we obtain the system of equations

$$\begin{cases} c_2 + c_3 = 0 \\ e^2c_1 + e^2c_2 + ec_3 = 0 \\ 2e^4c_1 + e^4c_2 + e^2c_3 = 0 \end{cases}$$

It's straightforward to verify that the only solution to this system is the trivial solution: $(c_1, c_2, c_3) = (0, 0, 0)$. Since $c_1\varphi + c_2\psi + c_3\xi = 0$ on $(-\infty, \infty)$ implies that $c_1 = c_2 = c_3 = 0$, we conclude that φ, ψ, and ξ are linearly independent on $(-\infty, \infty)$.

5 Auxiliary equation is $2r^2+7r-15=0$, which has roots $r_1 = 3/2$ and $r_2 = -5$. The general solution is therefore

$$y(t) = c_1 e^{3t/2} + c_2 e^{-5t},$$

and hence

$$y'(t) = \tfrac{3}{2} c_1 e^{3t/2} - 5c_2 e^{-5t}.$$

From the initial conditions $y(0) = -2$ and $y'(0) = 4$ we have

$$\begin{cases} c_1 + c_2 = -2 \\ \tfrac{3}{2}c_1 - 5c_2 = 4 \end{cases}$$

Solving this system yields $c_1 = -12/13$ and $c_2 = -14/13$, and so the solution to the initial value problem is

$$y(t) = -\tfrac{12}{13} e^{3t/2} - \tfrac{14}{13} e^{-5t}.$$

6 Auxiliary equation is $9r^2-12r+4=0$, which has double root $r = 2/3$. The general solution is therefore

$$y(t) = c_1 e^{2t/3} + c_2 t e^{2t/3}.$$

7 Auxiliary equation is $12r^3 - 28r^2 - 3r + 7 = 0$, which factors by grouping:

$$(3r-7)(2r-1)(2r+1) = 0.$$

Roots are $r = 7/3, 1/2, -1/2$. General solution:

$$y(t) = c_1 e^{-t/2} + c_2 e^{t/2} + c_3 e^{7t/3}.$$

8 Auxiliary equation is $r^2 + 9 = 0$, which has roots $r = \pm 3i$. Thus we have

$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$

With the initial conditions $y(0) = 1$ and $y'(0) = 1$ we find that $c_1 = 1$ and $c_2 = 1/3$, and so

$$y(t) = \cos 3t + \tfrac{1}{3} \sin 3t.$$

9a The nonhomogeneity is

$$f(t) = P_m(t) e^{\alpha t} = 4t^2,$$

so $m = 2$ and $\alpha = 0$. We see that α is not a root of the auxiliary equation $r^2 - 3r + 2 = 0$, so $s = 0$ and we have

$$y_p(t) = t^s e^{\alpha t} \sum_{k=0}^{m} A_k t^k = \sum_{k=0}^{2} A_k t^k = A_0 + A_1 t + A_2 t^2.$$

For convenience we may write $y_p(t) = A + Bt + Ct^2$, so that

$$y_p'(t) = B + 2Ct \quad \text{and} \quad y_p''(t) = 2C.$$

Substituting into the ODE yields

$$2C - 3(B + 2Ct) + 2(A + Bt + Ct^2) = 4t^2,$$

or

$$(2C)t^2 + (-6C + 2B)t + (2C - 3B + 2A) = 4t^2.$$

Equating coefficients gives the system

$$\begin{cases} 2C = 4 \\ 2B - 6C = 0 \\ 2A - 3B + 2C = 0 \end{cases}$$

which has solution $(A, B, C) = (7, 6, 2)$, and thus a particular solution to the ODE is $y_p(t) = 2t^2 + 6t + 7$.

9b The nonhomogeneity is

$$f(t) = P_m(t)e^{\alpha t} \sin \beta t = e^t \sin t,$$

so $m = 0$, $\alpha = 1$, and $\beta = 1$. We see that $\alpha + i\beta = 1 + i$ is not a root of the auxiliary equation $r^2 - 3r + 2 = 0$, so $s = 0$ and we have

$$y_p(t) = e^t \cos t \sum_{k=0}^{0} A_k t^k + e^t \sin t \sum_{k=0}^{0} B_k t^k = A_0 e^t \cos t + B_0 e^t \sin t,$$

or simply $y_p(t) = Ae^t \cos t + Be^t \sin t$. Now,

$$y_p'(t) = (-A + B)e^t \sin t + (A + B)e^t \cos t \quad \text{and} \quad y_p''(t) = 2Be^t \cos t - 2Ae^t \sin t,$$

and substitution into the ODE yields

$$e^t(2B \cos t - 2A \sin t) - 3e^t[(B - A) \sin t + (A + B) \cos t] + 2e^t(A \cos t + B \sin t) = e^t \sin t,$$

or simply

$$(A - B) \sin t + (-A - B) \cos t = \sin t.$$

Equating coefficients yields the system

$$\begin{cases} A - B = 1 \\ -A - B = 0 \end{cases}$$

which has solution $(A, B) = \left(\frac{1}{2}, -\frac{1}{2}\right)$, and thus a particular solution to the ODE is $y_p(t) = e^t \left(\frac{1}{2} \cos t - \frac{1}{2} \sin t\right)$.

9c Using the results of parts (a) and (b), by the Principle of Superposition a particular solution is

$$y_p(t) = \tfrac{1}{2}e^t(\cos t - \sin t) + 2t^2 + 6t + 7.$$

9d The auxiliary equation of the corresponding homogeneous equation $y'' - 3y' + 2y = 0$ is $r^2 - 3r + 2 = 0$, which has roots $1, 2$. Thus the general solution to the homogeneous equation is $y_h(t) = c_1e^t + c_2e^{2t}$, and by the Superposition Principle the general solution to $y'' - 3y' + 2y = e^t \sin t + 4t^2$ is

$$y(t) = \tfrac{1}{2}e^t(\cos t - \sin t) + 2t^2 + 6t + 7 + c_1e^t + c_2e^{2t}.$$

10a By the Method of Variation of Parameters we obtain the particular solution

$$y_p(t) = \frac{(2\ln t - 3)t^2e^{-2t}}{4}.$$

(See exercise #4.6.7 in the textbook.)

10b The auxiliary equation of the corresponding homogeneous equation $y'' + 4y' + 4y = 0$ is $r^2 + 4r + 4 = 0$, which has double root $r = -2$. The general solution of the homogeneous equation is thus

$$y_h(t) = c_1e^{-2t} + c_2te^{-2t},$$

and therefore the general solution to the original nonhomogeneous equation is, by the Superposition Principle,

$$y(t) = \frac{(2\ln t - 3)t^2e^{-2t}}{4} + c_1e^{-2t} + c_2te^{-2t}.$$

10c The factor $\ln(t)$ in the nonhomogeneity $e^{-2t}\ln(t)$ is not a polynomial, exponential, sine, or cosine function.