# Math 250 Exam #1 Key (Summer 2013)

**1** $T'(t) = k[M(t) - T(t)]$, or simply $T' = k(M - T)$, where $k$ is the constant of proportionality (not an arbitrary constant). Note: we could also write $T' = k(T - M)$, which in practice would result merely in the constant of proportionality $k$ reversing sign.

**2** We have

$$f(x, y) = 3x - \sqrt[3]{y-1},$$

which is continuous everywhere. However

$$f_y(x, y) = -\frac{1}{3}(y-1)^{-2/3} = -\frac{1}{3\sqrt[3]{(y-1)^2}}$$

is not continuous on the line $y = 1$ since the function is not defined there. The initial point $(2, 1)$ lies on this line, and so the Existence-Uniqueness Theorem does not imply that the initial value problem has a unique solution.

**3** Substitute $2e^{3t} - e^{2t}$ for $\theta$ in the equation to obtain

$$\begin{aligned}
(2e^{3t} - e^{2t})'' - (2e^{3t} - e^{2t})(2e^{3t} - e^{2t})' + 3(2e^{3t} - e^{2t}) &= -2e^{2t} \\
(18e^{3t} - 4e^{2t}) - (2e^{3t} - e^{2t})(6e^{3t} - 2e^{2t}) + (6e^{3t} - 3e^{2t}) &= -2e^{2t} \\
18e^{3t} - 4e^{2t} - 12e^{6t} + 10e^{5t} - 2e^{4t} + 6e^{3t} - 3e^{2t} &= -2e^{2t} \\
-12e^{6t} + 10e^{5t} - 2e^{4t} + 24e^{3t} - 7e^{2t} &= -2e^{2t}
\end{aligned}$$

The last equation is not true for all $t$ on *any* interval $I \subseteq \mathbb{R}$, and so the function $2e^{3t} - e^{2t}$ is *not* a solution to the ODE.

**4** Substitute $\varphi(x) = e^{mx}$ for $y$ to obtain

$$\begin{aligned}
2(e^{mx})''' + 9(e^{mx})'' - 5(e^{mx})' &= 0 \\
2m^3e^{mx} + 9m^2e^{mx} - 5me^{mx} &= 0 \\
(2m^3 + 9m^2 - 5m)e^{mx} &= 0
\end{aligned}$$

To satisfy the equation for all $x$ in some interval $I \subseteq \mathbb{R}$, it will be necessary to have

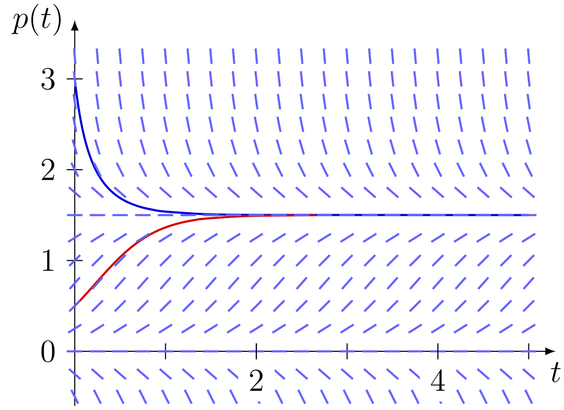
$$2m^3 + 9m^2 - 5m = 0.$$

Solving this equation for $m$, we have

$$m(2m-1)(m+5) = 0$$

and thus $m = 0, 1/2, -5$. This shows that $\varphi_1(x) = 1$, $\varphi_2(x) = e^{x/2}$, and $\varphi_3(x) = e^{-5x}$ are solutions to the ODE.

**5** The solution curves corresponding to the initial conditions $p(0) = 3$ and $p(0) = 0.5$ are below. If $p(0) = 2$ we have $p(t) \to 1.5^+$ as $t \to \infty$, so a population of 2000 can never be 500.




**6** We are given $(x_0, y_0) = (1, 0)$ and $h = 0.1$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_n$ | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
| $y_n$ | 0.0000 | 0.1000 | 0.2090 | 0.3246 | 0.4441 | 0.5644 |

**7** The equation is separable since

$$\frac{e^{x+y}}{y-1} = \frac{e^x e^y}{y-1} = e^x \cdot \frac{e^y}{y-1}.$$

Thus we have

$$\int \frac{y-1}{e^y} dy = \int e^x\, dx,$$

which yields

$$e^x + ye^{-y} = c$$

for arbitrary constant $c$.

**8** Writing the equation as $y' = 2\cos x\sqrt{y+1}$, we see the equation is separable. We get

$$\int \frac{1}{\sqrt{y+1}}\, dy = \int 2\cos x\, dx.$$

This integrates easily to give

$$2\sqrt{y+1} = 2\sin x + c.$$

Now, $y(\pi) = 0$ implies that

$$2\sqrt{0+1} = 2\sin\pi + c,$$

or $c = 2$. The (implicit) solution to the IVP is thus $2\sqrt{y+1} = 2\sin x + 2$, or $y = (\sin x + 1)^2 - 1$.

**9** The equation may be written as

$$y' + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x,$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3.$$

Multiplying the ODE by $x^3$ gives

$$x^3y' + 3x^2y = x\sin x - 3x^4,$$

which becomes $(x^3y)' = x\sin x - 3x^4$ and thus

$$x^3y = \int x\sin x\,dx - \frac{3}{5}x^5 + c.$$

Integration by parts gives

$$\int x\sin x\,dx = \sin x - x\cos x,$$

so that $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + c$ and therefore

$$y(x) = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + c\right).$$

is the general solution.

**10** The equation may be written as

$$x' + \frac{3}{t}x = t^2\ln t + \frac{1}{t^2},$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/t\,dt} = t^3.$$

Multiplying the ODE by $t^3$ gives $t^3x' + 3t^2x = t^5\ln t + t$, which becomes $(t^3x)' = t^5\ln t + t$ and thus

$$t^3x = \int t^5\ln t\,dt + \frac{1}{2}t^2 + c.$$

By integration by parts we find that

$$\int t^5\ln t\,dt = \frac{1}{6}t^6\ln t - \int \frac{1}{6}t^5\,dt = \frac{t^6}{6}\ln t - \frac{t^6}{36},$$

and so we have

$$t^3x = \frac{t^6}{6}\ln t - \frac{t^6}{36} + \frac{1}{2}t^2 + c.$$

Letting $t = 1$ and $x = 0$ (the initial condition) gives $0 = -\frac{1}{36} + \frac{1}{2} + c$, so that $c = -\frac{17}{36}$ and we obtain

$$x(t) = \frac{1}{6}t^3\left(\ln t - \frac{1}{6}\right) + \frac{1}{2t} - \frac{17}{36t^3}$$

as the solution to the IVP.

**11** Since the equation is exact there exists a function $F(x, y)$ such that

$$F_x(x, y) = \cos x \cos y + 2x \quad \text{and} \quad F_y(x, y) = -\sin x \sin y - 2y. \tag{1}$$

Integrate the first equation in (1) with respect to $x$ to get

$$F(x, y) = \int (\cos x \cos y + 2x)dx + g(y) = \sin x \cos y + x^2 + g(y). \tag{2}$$

Differentiating this with respect to $y$ yields

$$F_y(x, y) = -\sin x \sin y + g'(y),$$

and so using the second equation in (1) we obtain

$$-\sin x \sin y - 2y = -\sin x \sin y + g'(y),$$

or simply $g'(y) = -2y$. Hence $g(y) = -y^2 + c_1$ for some arbitrary constant $c_1$, and so (2) becomes

$$F(x, y) = \sin x \cos y + x^2 - y^2 + c_1.$$

The general implicit solution to the ODE is therefore

$$\sin x \cos y + x^2 - y^2 + c_1 = c_2$$

for arbitrary $c_2$, which we can write simply as

$$\sin x \cos y + x^2 - y^2 = c$$

by consolidating the arbitrary constants $c_1$ and $c_2$.

**12** We have

$$\frac{M_y - N_x}{N}(x) = \frac{2 - 4xy}{2x^2y - x} = -\frac{2}{x},$$

so

$$\mu(x) = \exp\left(\int -\frac{2}{x}dx\right) = e^{-2\ln x} = \frac{1}{x^2}.$$

Multiplying the ODE by $\mu(x)$ yields the exact equation

$$(3 + x^{-2}y) + (2y - x^{-1})y' = 0.$$

There exists a function $F$ such that

$$F_x(x, y) = 3 + x^{-2}y \quad \text{and} \quad F_y(x, y) = 2y - x^{-1}.$$

From the former equation comes

$$F(x, y) = 3x - \frac{y}{x} + g(y),$$

so the latter equation implies

$$-x^{-1} + g'(y) = 2y - x^{-1} \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + c_1,$$

$c_1$ arbitrary. The general solution to the ODE is $F(x, y) = c_2$, where $c_2$ is arbitrary. That is,

$$3x - \frac{y}{x} + y^2 = c,$$

where $c$ is the arbitrary constant deriving from $c_2 - c_1$.

**13** Multiply the ODE by $x^m y^n$:

$$(x^{m+3}y^{n+2} - 2x^m y^{n+3}) + (x^{m+4}y^{n+1})y' = 0.$$

For exactness we need $M_y = N_x$, or

$$(n+2)x^{m+3}y^{n+1} - 2(n+3)x^m y^{n+2} = (m+4)x^{m+3}y^{n+1}.$$

Matching coefficients of like terms, we find that we must have $n+2 = m+4$ and $-2(n+3) = 0$, which solves to give $m = -5$ and $n = -3$. Thus an integrating factor is $\mu(x,y) = x^{-5}y^{-3}$, and the ODE becomes

$$(x^{-2}y^{-1} - 2x^{-5}) + (x^{-1}y^{-2})y' = 0,$$

which is exact. We now find a function $F$ such that $F_x(x,y) = x^{-2}y^{-1} - 2x^{-5}$ and $F_y(x,y) = x^{-1}y^{-2}$. From the former equation we obtain

$$F(x,y) = \int \left(x^{-2}y^{-1} - 2x^{-5}\right) dx = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + g(y),$$

and from the latter equation comes

$$x^{-1}y^{-2} = F_y(x,y) = x^{-1}y^{-2} + g'(y),$$

or $g'(y) = 0$. Thus $g(y) = c_1$ for some constant $c_1$, and we have

$$F(x,y) = -x^{-1}y^{-1} + \tfrac{1}{2}x^{-4} + c_1.$$

The implicit solution to the ODE is therefore $F(x,y) = c_2$ for arbitrary constant $c_2$, which we may write simply as

$$\tfrac{1}{2}x^{-4} - x^{-1}y^{-1} = c$$

by merging constant terms. Another solution is $y \equiv 0$.