Math 250 Exam #4 Key (Summer 2012)

1. $y(t) = \cos t - 4e^{5t} + 8e^{2t}$.

2a. We have $g(t) = 20u(t) - [20 - 20u(3\pi - t)] + [20 - 20u(4\pi - t)]$, which simplifies to

$$g(t) = 20[u(t) + u(3\pi - t) - u(4\pi - t)],$$

or even

$$g(t) = 20[1 + u(3\pi - t) - u(4\pi - t)]$$

if $t \geq 0$ is understood.

2b. First we need

$$\mathcal{L}[u(a-t)](s) = \int_0^\infty e^{-st}u(a-t)\,dt = \int_0^a e^{-st}\,dt = \frac{1-e^{-as}}{s}.$$

Now,

$$\begin{aligned}\mathcal{L}[g(t)](s) &= 20\mathcal{L}[1](s) + 20\mathcal{L}[u(3\pi - t)](s) - 20\mathcal{L}[u(4\pi - t)](s)\\ &= \frac{20}{s} + 20\left(\frac{1-e^{-3\pi s}}{s}\right) - 20\left(\frac{1-e^{-4\pi s}}{s}\right).\end{aligned}$$

2c. Letting $\mathcal{I}(s) = \mathcal{L}[I(t)](s)$, we have

$$\begin{aligned}[s^2\mathcal{I} - sI(0) - I'(0)] + 2[s\mathcal{I} - I(0)] + 2\mathcal{I} &= \frac{20}{s} + 20\left(\frac{1-e^{-3\pi s}}{s}\right) - 20\left(\frac{1-e^{-4\pi s}}{s}\right)\\ s^2\mathcal{I} - 10s + 2s\mathcal{I} - 20 + 2\mathcal{I} &= \frac{20 - 20e^{-3\pi s} + 20e^{-4\pi s}}{s}\\ (s^2+2s+2)\mathcal{I} &= 10s + 20 + \frac{20 - 20e^{-3\pi s} + 20e^{-4\pi s}}{s},\end{aligned}$$

and thus

$$\mathcal{I}(s) = \frac{10s+20}{s^2+2s+2} + \frac{20}{s(s^2+2s+2)} - \frac{20}{s(s^2+2s+2)}e^{-3\pi s} + \frac{20}{s(s^2+2s+2)}e^{-4\pi s}. \tag{1}$$

Partial fraction decomposition gives

$$\frac{20}{s(s^2+2s+2)} = \frac{10}{s} - \frac{10s+20}{s^2+2s+2},$$

and so (1) becomes

$$\mathcal{I}(s) = \frac{10}{s} - \left(\frac{10}{s} - \frac{10s+20}{s^2+2s+2}\right)e^{-3\pi s} + \left(\frac{10}{s} - \frac{10s+20}{s^2+2s+2}\right)e^{-4\pi s}.$$

Rewriting this as

$$\mathcal{I}(s) = \frac{10}{s} - 10\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-3\pi s}$$

$$+ 10\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-4\pi s}$$

and taking the inverse Laplace transform of both sides, we obtain

$$I(t) = 10 - 10u(t-3\pi)\left[1 - e^{3\pi - t}\cos(t-3\pi) - e^{3\pi-t}\sin(t-3\pi)\right] \\ + 10u(t-4\pi)\left[1 - e^{4\pi-t}\cos(t-4\pi) - e^{4\pi-t}\sin(t-4\pi)\right],$$

or equivalently

$$I(t) = 10 - 10u(t-3\pi)\big[1 + (\cos t + \sin t)e^{-(t-3\pi)}\big] + 10u(t-4\pi)\big[1 - (\cos t + \sin t)e^{-(t-4\pi)}\big].$$

3a. Trivial.

3b. Trivial.

4.

5. Solution will be of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Substituting this into the ODE gives

$$\sum_{k=1}^{\infty} k c_k x^{k-1} - \sum_{k=0}^{\infty} c_k x^k = 0.$$

Reindexing, we obtain

$$\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k = 0,$$

or equivalently

$$\sum_{k=0}^{\infty}\left[(k+1)c_{k+1} - c_k\right]x^k = 0.$$

This implies that $(k+1)c_{k+1} - c_k = 0$ for all $k \geq 0$, or $c_{k+1} = c_k/(k+1)$. From this we find that $c_1 = c_0$, $c_2 = c_1/2 = c_0/2!$, $c_3 = c_2/3 = c_0/3!$, and in general $c_k = c_0/k!$. Therefore

$$y(x) = \sum_{k=0}^{\infty}\frac{c_0}{k!}x^k = c_0 + c_0 x + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \cdots,$$

where c_0 is an arbitrary constant.

6. We find a general solution of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k,$$

with the series converging on some open interval I containing 0. Substituting this into the ODE yields

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - x^2 \sum_{k=1}^{\infty} kc_k x^{k-1} - x \sum_{k=0}^{\infty} c_k x^k = 0,$$

and thus

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=1}^{\infty} kc_k x^{k+1} - \sum_{k=0}^{\infty} c_k x^{k+1} = 0.$$

Reindexing so that all series feature x^k, we have

$$\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k - \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0.$$

Finally we contrive to have the index of each series start at 2 by removing the first two terms of the leftmost series and the first term of the rightmost series:

$$\left[2c_2 + 6c_3 x + \sum_{k=2}^{\infty}(k+1)(k+2)c_{k+2}x^k\right] - \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k - \left[c_0 x + \sum_{k=2}^{\infty} c_{k-1}x^k = 0\right].$$

Hence

$$2c_2 + (6c_3 - c_0)x + \sum_{k=2}^{\infty}\left[(k+1)(k+2)c_{k+2} - (k-1)c_{k-1} - c_{k-1}\right]x^k = 0,$$

which simplifies to become

$$2c_2 + (6c_3 - c_0)x + \sum_{k=2}^{\infty}\left[(k+1)(k+2)c_{k+2} - kc_{k-1}\right]x^k = 0.$$

This implies that $2c_2 = 0$, $6c_3 - c_0 = 0$, and

$$(k+1)(k+2)c_{k+2} - kc_{k-1} = 0$$

for all $k \geq 2$. That is, $c_2 = 0$, $c_3 = c_0/6 = c_0/(2 \cdot 3)$, and

$$c_{k+2} = \frac{k}{(k+1)(k+2)}c_{k-1}$$

for $k = 2, 3, 4, \ldots$. The recursion relation enables us to express all c_k exclusively in terms of c_0 and c_1:

$$c_4 = \frac{2}{3 \cdot 4}c_1 \qquad c_5 = \frac{3}{4 \cdot 5}c_2 = 0$$

$$c_6 = \frac{4}{5 \cdot 6}c_3 = \frac{4}{2 \cdot 3 \cdot 5 \cdot 6}c_0 \qquad c_7 = \frac{5}{6 \cdot 7}c_4 = \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7}c_1$$

$$c_8 = \frac{6}{7 \cdot 8}c_5 = 0 \qquad c_9 = \frac{7}{8 \cdot 9}c_6 = \frac{4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}c_0$$

$$c_{10} = \frac{8}{9 \cdot 10}c_7 = \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}c_1 \qquad c_{11} = \frac{9}{10 \cdot 11}c_8 = 0$$

$$c_{12} = \frac{10}{11 \cdot 12} c_9 = \frac{4 \cdot 7 \cdot 10}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 11 \cdot 12} c_0$$

So we have

$$y(x) = c_0 + c_1 x + \frac{c_0}{2 \cdot 3} x^3 + \frac{2c_1}{3 \cdot 4} x^4 + \frac{4c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{2 \cdot 5 c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{4 \cdot 7 c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \frac{2 \cdot 5 \cdot 8 c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \frac{4 \cdot 7 \cdot 10 c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 11 \cdot 12} x^{12} + \cdots$$

Setting $c_0 = 0$ and $c_1 = 1$ yields the particular solution

$$\begin{aligned} y_1(x) &= x + \frac{2}{3 \cdot 4} x^4 + \frac{2 \cdot 5}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{2 \cdot 5 \cdot 8}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \cdots \\ &= x + \frac{2^2}{4!} x^4 + \frac{2^2 \cdot 5^2}{7!} x^7 + \frac{2^2 \cdot 5^2 \cdot 8^2}{10!} x^{10} + \cdots \\ &= x + \sum_{k=1}^{\infty} \frac{2^2 \cdot 5^2 \cdots (3k-1)^2}{(3k+1)!} x^{3k+1}, \end{aligned}$$

and setting $c_0 = 1$ and $c_1 = 0$ yields the particular solution

$$\begin{aligned} y_2(x) &= 1 + \frac{1}{2 \cdot 3} x^3 + \frac{4}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{4 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \frac{4 \cdot 7 \cdot 10}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 11 \cdot 12} x^{12} + \cdots \\ &= 1 + \frac{1}{3!} x^3 + \frac{4^2}{6!} x^6 + \frac{4^2 \cdot 7^2}{9!} x^9 + \frac{4^2 \cdot 7^2 \cdot 10^2}{12!} x^{12} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{4^2 \cdot 7^2 \cdots (3k-2)^2}{(3k)!} x^{3k}. \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, the general solution to the ODE may be expressed as

$$y(x) = a_0 \left[x + \sum_{k=1}^{\infty} \frac{2^2 \cdot 5^2 \cdots (3k-1)^2}{(3k+1)!} x^{3k+1} \right] + a_1 \left[1 + \sum_{k=1}^{\infty} \frac{4^2 \cdot 7^2 \cdots (3k-2)^2}{(3k)!} x^{3k} \right]$$

for all $x \in I$, where a_0 and a_1 are arbitrary constants.

7. Since $x = 2$ is an ordinary point for the ODE, we expect to find a general solution of the form

$$y(x) = \sum_{k=0}^{\infty} c_k (x-2)^k, \tag{2}$$

with the power series converging on some open interval I containing 2. From (2) comes

$$y'(x) = \sum_{k=1}^{\infty} k c_k (x-2)^{k-1}$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) c_k (x-2)^{k-2},$$

which when substituted into the ODE yields

$$x^2\sum_{k=2}^{\infty}k(k-1)c_k(x-2)^{k-2}-\sum_{k=1}^{\infty}kc_k(x-2)^{k-1}+\sum_{k=0}^{\infty}c_k(x-2)^k=0. \tag{3}$$

It will be expedient to express x^2 in terms of $x-2$. Since $(x-2)^2=x^2-4x+4$ we have

$$x^2=(x-2)^2+4x-4=(x-2)^2+4(x-2)+4,$$

and so (3) becomes

$$[(x-2)^2+4(x-2)+4]\sum_{k=2}^{\infty}k(k-1)c_k(x-2)^{k-2}-\sum_{k=1}^{\infty}kc_k(x-2)^{k-1}+\sum_{k=0}^{\infty}c_k(x-2)^k=0,$$

and thus

$$\sum_{k=2}^{\infty}k(k-1)c_k(x-2)^k+4\sum_{k=2}^{\infty}k(k-1)c_k(x-2)^{k-1}+4\sum_{k=2}^{\infty}k(k-1)c_k(x-2)^{k-2}$$
$$-\sum_{k=1}^{\infty}kc_k(x-2)^{k-1}+\sum_{k=0}^{\infty}c_k(x-2)^k=0.$$

Adding zero terms and reindexing where needed, we obtain

$$\sum_{k=0}^{\infty}k(k-1)c_k(x-2)^k+4\sum_{k=0}^{\infty}k(k+1)c_{k+1}(x-2)^k+4\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}(x-2)^k$$
$$-\sum_{k=0}^{\infty}(k+1)c_{k+1}(x-2)^k+\sum_{k=0}^{\infty}c_k(x-2)^k=0,$$

or equivalently

$$\sum_{k=0}^{\infty}\left[k(k-1)c_k+4k(k+1)c_{k+1}+4(k+1)(k+2)c_{k+2}-(k+1)c_{k+1}+c_k\right](x-2)^k=0$$

for all $x\in I$. Therefore we have

$$k(k-1)c_k+4k(k+1)c_{k+1}+4(k+1)(k+2)c_{k+2}-(k+1)c_{k+1}+c_k=0$$

for all $k\geq 0$, which rearranges to become

$$c_{k+2}=-\frac{(4k^2+3k-1)c_{k+1}+(k^2-k+1)c_k}{4k^2+12k+8}. \tag{4}$$

Using the recursion relation (4), we obtain

$$c_2=\frac{c_1-c_0}{8}$$

and

$$c_3=-\frac{6c_2+c_1}{24}=-\frac{1}{4}\left(\frac{c_1-c_0}{8}\right)-\frac{1}{24}c_1=\frac{3c_0-7c_1}{96}.$$

Hence

$$y(x)=c_0+c_1(x-2)+c_2(x-2)^2+c_3(x-2)^3+\cdots$$

$$= c_0 + c_1(x-2) + \frac{c_1 - c_0}{8}(x-2)^2 + \frac{3c_0 - 7c_1}{96}(x-2)^3 + \cdots$$

is a power series expansion about 2 for a general solution to the ODE.