Math 250 Exam #3 Key (Summer 2012)

1. The model for the mass-spring system is $0.25y'' + 2y' + 8y = 0$, $y(0) = -0.50$, $y'(0) = -2$. The auxiliary equation is $0.25r^2+2r+8 = 0$, or $r^2+8r+32 = 0$, which has roots $\alpha \pm i\beta = -4 \pm 4i$. Thus the general solution to the ODE is

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t).$$

From the initial condition $y(0) = -0.50$ comes $c_1 = -0.50$, and from

$$y'(t) = -4e^{-4t}(c_1 \cos 4t + c_2 \sin 4t) + e^{-4t}(-4c_1 \sin 4t + 4c_2 \cos 4t) \tag{1}$$

and the initial condition $y'(0) = -2$ comes

$$-4e^0(c_1 \cos 0 + c_2 \sin 0) + e^0(-4c_1 \sin 0 + 4c_2 \cos 0) = -2,$$

which simplifies as $-4c_1 + 4c_2 = -2$ and finally $c_2 = -1$. Therefore

$$y(t) = -e^{-4t}(0.50 \cos 4t + \sin 4t).$$

The global minimum value of the function $y(t)$ will be $t^* = \min\{t \in [0, \infty) : y'(t) = 0\}$. Setting $y'(t) = 0$, from (1) we obtain

$$-4e^{-4t}(-0.50 \cos 4t - \sin 4t) + e^{-4t}(2 \sin 4t - 4 \cos 4t) = 0,$$

which becomes $6 \sin 4t - 2 \cos 4t = 0$, so that $\tan 4t = 1/3$ and finally

$$t = \frac{1}{4}\left[\arctan\left(\frac{1}{3}\right) + n\pi\right], \quad n \in \mathbb{Z}.$$

Setting $n = 0$ will give

$$t^* = \frac{1}{4}\arctan\left(\frac{1}{3}\right) \approx 0.08.$$

That is, the object attains its maximum displacement to the left at time 0.08 sec.

2. Equation of motion is

$$y(t) \approx e^{-1.25t}(0.04689 \cos 6.89t + 0.00848 \sin 6.89t) + 0.00311 \cos t + 0.00016 \sin t.$$

3. Applying the integration by parts technique,

$$\begin{aligned}
\mathcal{L}[F](s) &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(1-t)\, dt \\
&= \left[-\frac{1-t}{s}e^{-st}\right]_0^1 - \int_0^1 \frac{1}{s}e^{-st}\, dt = \left[\frac{t-1}{s}e^{-st}\right] + \frac{1}{s^2}\left[e^{-st}\right]_0^1 \\
&= \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right) = \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{1}{s^2}.
\end{aligned}$$

4. Using the table provided,

$$\mathcal{L}\big[t^2e^{5t}\big](s) = \frac{2!}{(s-5)^{2+1}} = \frac{2}{(s-5)^3}$$

5. Using the table provided,

$$\begin{aligned}\mathcal{L}\big[t^5 - 7e^{-3t}\sin 4t\big](s) &= \mathcal{L}\big[t^5\big](s) - 7\mathcal{L}\big[e^{-3t}\sin 4t\big](s)\\ &= \frac{5!}{(s-0)^{5+1}} - 7\cdot\frac{4}{(s+3)^2+4^2}\\ &= \frac{120}{s^6} - \frac{28}{(s+3)^2+16}\end{aligned}$$

6. Use a trigonometric identity for this, along with the table provided:

$$\begin{aligned}\mathcal{L}\big[e^{8t}\cos^2 t\big](s) &= \mathcal{L}\big[e^{8t}\cdot\big(\tfrac{1}{2}+\tfrac{1}{2}\cos 2t\big)\big](s) = \tfrac{1}{2}\mathcal{L}\big[e^{8t}\big](s) + \tfrac{1}{2}\mathcal{L}\big[e^{8t}\cos 2t\big](s)\\ &= \frac{1}{2}\cdot\frac{0!}{(s-8)^{0+1}} + \frac{1}{2}\cdot\frac{s-8}{(s-8)^2+2^2} = \frac{1}{2(s-8)} + \frac{s-8}{2(s-8)^2+8}\end{aligned}$$

7. Partial fraction decomposition is necessary: we have

$$\frac{7s^2-41s+84}{(s-1)(s^2-4s+13)} = \frac{5}{s-1} + \frac{2s-19}{s^2-4s+13},$$

and so

$$\begin{aligned}\mathcal{L}^{-1}[F](t) &= \mathcal{L}^{-1}\left[\frac{5}{s-1}\right](t) + \mathcal{L}^{-1}\left[\frac{2s-19}{s^2-4s+13}\right](t)\\ &= 5e^t + \mathcal{L}^{-1}\left[\frac{2s-19}{(s-2)^2+3^2}\right](t)\\ &= 5e^t + 2\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right](t) - 5\mathcal{L}^{-1}\left[\frac{3}{(s-2)^2+3^2}\right](t)\\ &= 5e^2 + 2e^{2t}\cos 3t - 5e^{2t}\sin 3t\end{aligned}$$