Math 250 Exam #2 Key (Summer 2012)

1. First suppose that $y > 0$. Then

$$y' = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{y} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{\sqrt{y^2}} = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^{-2} + 1}.$$

Letting $v = y/x$, the equation becomes $v + xv' = v + \sqrt{v^{-2}+1}$, which is separable and so becomes

$$\int \frac{1}{\sqrt{v^{-2}+1}}\,dv = \int \frac{1}{x}\,dx \tag{1}$$

Now, if $x > 0$, then $v > 0$ also, in which case

$$v\sqrt{v^{-2}+1} = \sqrt{v^2}\sqrt{v^{-2}+1} = \sqrt{v^2+1},$$

and so, making the substitution $u = v^2 + 1$ along the way, we get

$$\int \frac{1}{\sqrt{v^{-2}+1}}\,dv = \int \frac{v}{v\sqrt{v^{-2}+1}}\,dv = \int \frac{v}{\sqrt{v^2+1}}\,dv = \int \frac{1/2}{\sqrt{u}}\,du$$
$$= \sqrt{u} + c = \sqrt{v^2+1} + c = \sqrt{y^2/x^2+1} + c.$$

Putting this into (1) gives us the solution

$$\sqrt{\frac{y^2}{x^2}+1} = \ln|x| + c. \tag{2}$$

If $x < 0$, then $v < 0$ also, in which case

$$v\sqrt{v^{-2}+1} = -\sqrt{v^2}\sqrt{v^{-2}+1} = -\sqrt{v^2+1}.$$

Performing the same manipulations as before (only with a negative sign attached) results in the solution

$$-\sqrt{\frac{y^2}{x^2}+1} = \ln|x| + c. \tag{3}$$

Now, if we suppose that $y < 0$, much the same analysis is performed, again broken into the two cases $x > 0$ and $x < 0$. If $x > 0$, the solution (3) results, and if $x < 0$, the solution (2) results. If $\operatorname{sgn}(x)$ and $\operatorname{sgn}(y)$ denote the sign of x and y, respectively, then the general solution can be written as

$$\operatorname{sgn}(x)\operatorname{sgn}(y)\sqrt{y^2/x^2+1} = \ln|x| + c.$$

2. The Bernoulli equation has $n = 3$, $P(x) = -1$, and $Q(x) = e^{2x}$. Let $v = y^{-2}$, so that the equation becomes $v' + 2v = -2e^{2x}$ as indicated by the formula in the notes (and book). This is a linear equation with $P(x) = 2$ and $Q(x) = -2e^{2x}$, so an integrating factor is

$$\mu(x) = e^{\int 2\,dx} = e^{2x}.$$

Multiplying $v' + 2v = -2e^{2x}$ by e^{2x} gives $v'e^{2x} + 2ve^{2x} = -2e^{4x}$, which becomes $(ve^{2x})' = -2e^{4x}$, and so by integration we obtain

$$ve^{2x} = \int -2e^{4x}\,dx = -\frac{1}{2}e^{4x} + c.$$

Hence $v = -\frac{1}{2}e^{2x} + ce^{-2x}$, so that $y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}$. That is, the general solution to the ODE is

$$y^2 = \frac{2}{ce^{-2x} - e^{2x}}.$$

Another solution happens to be $y \equiv 0$.

3. Let $x(t)$ be the number of gallons of Cl in the pool at time t, so $x(0) = 1$ (0.01% of 10,000). Now, 5 gallons of solution that is 0.001% Cl by volume is coming in per minute, which is to say that 0.00005 gallons of Cl is entering per minute. Meanwhile there are $x(t)/10,000$ gallons of Cl per gallon of solution in the pool, and this solution is being pumped out at a rate of 5 gallons per minute. Thus, $5x(t)/10,000$ gallons of Cl is leaving per minute. We have

$$x'(t) = 0.00005 - \frac{5x(t)}{10,000} = \frac{0.1 - x}{2000}$$

This equation is separable, and so becomes

$$\int \frac{2000}{0.1 - x}\,dx = \int dt,$$

and hence $-2000 \ln|x - 0.1| = t + c$. Solving for x gives $x(t) = 0.1 + Ke^{-t/2000}$, and using the initial condition $x(0) = 1$ we find that $K = 0.9$. So finally we have $x(t) = 0.1 + 0.9e^{-t/2000}$.

The amount of Cl in the pool after 60 minutes (1 hour) is $x(60) = 0.1 + 0.9e^{-60/2000} = 0.973$ gallons, which means the pool is 0.00973% Cl.

We now find the time t when the pool is 0.002% Cl, or in other words $x(t) = 0.2$ gallons Cl. The equation is $0.2 = 0.1 + 0.9e^{-t/2000}$, which solves to give $t = 4394.4$ minutes, or 73.24 hours.

4. Auxiliary equation is $r^2 - 4r - 5 = 0$, so $r = -1, 5$ and the general solution is $y(t) = c_1e^{-t} + c_2e^{5t}$. Using the initial conditions $y(-1) = 3$ and $y'(-1) = 9$, we find that $c_1 = e^{-1}$ and $c_2 = 2e^5$. Thus the solution to the IVP is $y(t) = e^{-t-1} + 2e^{5t+5}$, or

$$y(t) = e^{-(t+1)} + 2e^{5(t+1)}$$

will also do.

5. Auxiliary equation is $r^3 - 6r^2 - r + 6 = 0$, which factors as $(r-6)(r^2-1) = 0$ and finally $(r-6)(r-1)(r+1) = 0$. Thus $r = 6, 1, -1$, and the general solution is

$$y(t) = c_1e^{6t} + c_2e^t + c_3e^{-t},$$

a three-parameter family of functions.

6. Auxiliary equation is $r^2 - 2r + 26 = 0$, which has roots $r = 1 \pm 5i$. So $\alpha = 1$ and $\beta = 5$, and the general solution is therefore $y(t) = e^t(c_1 \cos 5t + c_2 \sin 5t)$.

7a. The auxiliary equation $r^2+4=0$ has roots $r=\pm 2i$. The nonhomogeneity $f(t)=8\sin 2t$ has the form $P_m(t)e^{\alpha t}\sin\beta t$ with $m=0$, $\alpha=0$, and $\beta=2$. Since $\alpha+i\beta=2i$ is a root of the auxiliary equation, we take $s=1$ in the Method of Undetermined Coefficients, and so the form for y_p is $y_p(t)=At\cos 2t+Bt\sin 2t$. Substitution into the ODE gives

$$(At\cos 2t+Bt\sin 2t)''+4(At\cos 2t+Bt\sin 2t)=8\sin 2t,$$

which becomes

$$(-4A\sin 2t+4B\cos 2t-4At\cos 2t-4Bt\sin 2t)+(4At\cos 2t+4Bt\sin 2t)=8\sin 2t$$

and finally

$$-4A\sin 2t+4B\cos 2t=8\sin 2t.$$

From this we see we need $-4A=8$ and $4B=0$, giving us $A=-2$ and $B=0$. Hence $y_p(t)=-2t\cos 2t$.

7b. Since the roots of the auxiliary equation are $\pm 2i$, the homogeneous solution to the ODE is $y_h(t)=c_1\cos 2t+c_2\sin 2t$. Therefore the general solution is

$$y(t)=-2t\cos 2t+c_1\cos 2t+c_2\sin 2t$$

by superposition.

8a. Auxiliary equation $2r^2+3r+1=0$ has roots $-1/2,-1$. Start with $2y''+3y'+y=t^2$, whose nonhomogeneity t^2 is of the form $P_m(t)e^{\alpha t}$ with $m=2$ and $\alpha=0$. Since α is not a root of the auxiliary equation we set $s=0$ in the Method of Undetermined Coefficients to obtain $y_{p_1}(t)=At^2+Bt+C$ as the form of a particular solution. Substituting $y_{p_1}(t)$, $y'_{p_1}(t)=2At+B$, and $y''_{p_1}(t)=2A$ in $2y''+3y'+y=t^2$ gives

$$2(2A)+3(2At+B)+(At^2+Bt+C)=t^2,$$

which rearranges to become

$$At^2+(6A+B)t+(4A+3B+C)=t^2.$$

The system of equations

$$\begin{array}{ccccccc} A & & & & & = & 1 \\ 6A & + & B & & & = & 0 \\ 4A & + & 3B & + & C & = & 0 \end{array}$$

results, which has solution $(A,B,C)=(1,-6,14)$. Hence $y_{p_1}(t)=t^2-6t+14$.

Now consider $2y''+3y'+y=3\sin t$, whose nonhomogeneity $3\sin t$ is of the form $P_m(t)e^{\alpha t}\sin\beta t$ with $m=0$, $\alpha=0$, and $\beta=1$. Since $\alpha+i\beta=i$ is not a root of the auxiliary equation we set $s=0$ to get $y_{p_2}(t)=A\cos t+B\sin t$ as the form of a particular solution. Substituting $y_{p_2}(t)$, $y'_{p_2}(t)=-A\sin t+B\cos t$, and $y''_{p_2}(t)=-A\cos t-B\sin t$ in $2y''+3y'+y=3\sin t$ gives

$$2(-A\cos t-B\sin t)+3(-A\sin t+B\cos t)+(A\cos t+B\sin t)=3\sin t,$$

which a little algebra renders as

$$(-A+3B)\cos t+(-3A-B)\sin t=3\sin t.$$

The system of equations

$$\begin{array}{rcrcl} -A & + & 3B & = & 0 \\ -3A & - & B & = & 3 \end{array}$$

results, which has solution $(A, B) = \left(-\frac{9}{10}, -\frac{3}{10}\right)$. Hence $y_{p_2}(t) = -\frac{9}{10}\cos t - \frac{3}{10}\sin t$.

By the Superposition Principle a particular solution to the original ODE is

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) = t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t.$$

8b. The general solution to $2y'' + 3y' + y = 0$ is $y_h(t) = c_1e^{-t} + c_2e^{-t/2}$. Therefore the general solution to $2y'' + 3y' + y = t^2 + 3\sin t$ is

$$y(t) = y_p(t) + y_h(t) = t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t + c_1e^{-t} + c_2e^{-t/2}.$$