Math 250 Exam #1 Key (Summer 2012)

1. $T'(t) = k[M(t) - T(t)]$, or simply $T' = k(M - T)$.

2. Second-order ordinary nonlinear differential equation with independent variable t and dependent variable y.

3. Substitute $2e^{3t} - e^{2t}$ for θ in the equation to obtain

$$\begin{aligned}(2e^{3t} - e^{2t})'' - (2e^{3t} - e^{2t})(2e^{3t} - e^{2t})' + 3(2e^{3t} - e^{2t}) &= -2e^{2t}\\ (18e^{3t} - 4e^{2t}) - (2e^{3t} - e^{2t})(6e^{3t} - 2e^{2t}) + (6e^{3t} - 3e^{2t}) &= -2e^{2t}\\ 18e^{3t} - 4e^{2t} - 12e^{6t} + 10e^{5t} - 2e^{4t} + 6e^{3t} - 3e^{2t} &= -2e^{2t}\\ -12e^{6t} + 10e^{5t} - 2e^{4t} + 24e^{3t} - 7e^{2t} &= -2e^{2t}\end{aligned}$$

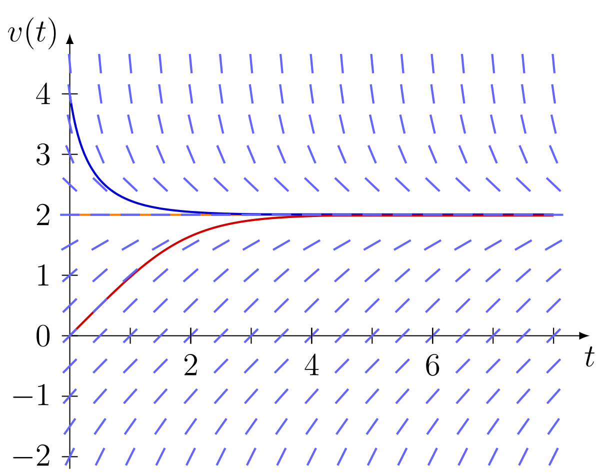
The last equation is not true for all t on *any* interval $I \subseteq \mathbb{R}$, and so the function $2e^{3t} - e^{2t}$ is *not* a solution to the ODE.

4. Substitute $\varphi(x) = e^{mx}$ for y to obtain

$$\begin{aligned}(e^{mx})''' + 3(e^{mx})'' + 2(e^{mx})' &= 0\\ m^3e^{mx} + 3m^2e^{mx} + 2me^{mx} &= 0\\ (m^3 + 3m^2 + 2m)e^{mx} &= 0\end{aligned}$$

To satisfy the equation for all x in some interval $I \subseteq \mathbb{R}$, it will be necessary to have $m^3 + 3m^2 + 2m = 0$. Solving this equation for m, we have: $m(m+1)(m+2) = 0$ and thus $m = 0, -1, -2$. This show that $\varphi_1(x) = 1$, $\varphi_2(x) = e^{-x}$, and $\varphi_3(x) = e^{-2x}$ are solutions to the ODE.

5. The solution curves corresponding to the initial conditions $v(0) = 0$, $v(0) = 2$, and $v(0) = 4$ are below. It can be seen that $v(t) \to 2$ as $t \to \infty$.



6. We are given $(x_0, y_0) = (1, 0)$ and $h = 0.1$.

n	0	1	2	3	4	5
x_n	1.0	1.1	1.2	1.3	1.4	1.5
y_n	0.0000	0.1000	0.2090	0.3246	0.4441	0.5644

7. The equation is separable:

$$x'(t) = \frac{e^{2x}}{x} \cdot \frac{t}{e^t}.$$

We obtain

$$\int xe^{2x}\,dx = \int te^{-t}\,dt,$$

which by integration by parts becomes

$$\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} = -te^{-t} - e^{-t} + c.$$

Multiplying by 4 and rearranging then gives a one-parameter family of implicit solutions to the ODE: $(2x-1)e^{2x} + 4e^{-t}(t+1) = c$.

8. Writing the equation as $y' = 2\cos x\sqrt{y+1}$, we see the equation is separable. We get

$$\int \frac{1}{\sqrt{y+1}}\,dy = \int 2\cos x\,dx.$$

This integrates easily to give $2\sqrt{y+1} = 2\sin x + c$. Now, $y(\pi) = 0$ implies that $2\sqrt{0+1} = 2\sin\pi + c$, or $c = 2$. The (implicit) solution to the IVP is thus $2\sqrt{y+1} = 2\sin x + 2$, or $y = (\sin x + 1)^2 - 1$.

9. The equation may be written as $y' + \dfrac{3}{x}y = \dfrac{\sin x}{x^2} - 3x$, which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln x} = x^3.$$

Multiplying the ODE by x^3 gives $x^3y' + 3x^2y = x\sin x - 3x^4$, which becomes $(x^3y)' = x\sin x - 3x^4$ and thus

$$x^3y = \int x\sin x\,dx - \frac{3}{5}x^5 + c.$$

Integration by parts gives

$$\int x\sin x\,dx = \sin x - x\cos x,$$

so that $x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + c$ and therefore

$$y(x) = \frac{1}{x^3}\left(\sin x - x\cos x - \frac{3}{5}x^5 + c\right).$$

is the general solution.

10. The equation may be written as

$$x' + \frac{3}{t}x = t^2 \ln t + \frac{1}{t^2},$$

which is the standard form for a 1st-order linear ODE. An integrating factor is

$$\mu(x) = e^{\int 3/t\, dt} = t^3.$$

Multiplying the ODE by t^3 gives $t^3 x' + 3t^2 x = t^5 \ln t + t$, which becomes $(t^3 x)' = t^5 \ln t + t$ and thus

$$t^3 x = \int t^5 \ln t \, dt + \frac{1}{2}t^2 + c.$$

By integration by parts we find that

$$\int t^5 \ln t \, dt = \frac{1}{6}t^6 \ln t - \int \frac{1}{6}t^5 \, dt = \frac{t^6}{6}\ln t - \frac{t^6}{36},$$

and so we have

$$t^3 x = \frac{t^6}{6}\ln t - \frac{t^6}{36} + \frac{1}{2}t^2 + c.$$

Letting $t = 1$ and $x = 0$ (the initial condition) gives $0 = -\frac{1}{36} + \frac{1}{2} + c$, so that $c = -\frac{17}{36}$ and we obtain

$$x(t) = \frac{1}{6}t^3\left(\ln t - \frac{1}{6}\right) + \frac{1}{2t} - \frac{17}{36t^3}$$

as the solution to the IVP.

11. Since the equation is exact, there exists a function $F(x,y)$ such that

$$F_x(x,y) = 2x + \frac{y}{1+x^2y^2} \quad \text{and} \quad F_y(x,y) = \frac{x}{1+x^2y^2} - 2y.$$

Integrate the first equation with respect to x:

$$\begin{aligned} F(x,y) &= \int \left(2x + \frac{y}{1+x^2y^2}\right)dx + g(y) = x^2 + \frac{1}{y}\int \left(\frac{1}{(1/y)^2 + x^2}\right)dx + g(y) \\ &= x^2 + \frac{1}{y}\cdot\frac{1}{1/y}\arctan\left(\frac{x}{1/y}\right) + g(y) = x^2 + \arctan(xy) + g(y). \end{aligned}$$

Now from the second equation we have

$$\frac{\partial}{\partial y}\left[x^2 + \arctan(xy) + g(y)\right] = \frac{x}{1+x^2y^2} - 2y,$$

and thus

$$\frac{x}{1+(xy)^2} + g'(y) = \frac{x}{1+x^2y^2} - 2y.$$

From this we obtain $g'(y) = -2y$, and finally $g(y) = -y^2 + c_1$. Thus

$$F(x,y) = x^2 + \arctan(xy) - y^2 + c_1.$$

The implicit solution to the ODE is of the form $F(x,y) = c_2$; that is,

$$x^2 + \arctan(xy) - y^2 + c_1 = c_2,$$

or simply

$$x^2 - y^2 + \arctan(xy) = c$$

if we combine the arbitrary constants c_1 and c_2.

12. Here $M(x,y) = y^2 + 2xy$ and $N(x,y) = -x^2$. Now, since

$$\frac{N_x - M_y}{M} = \frac{-2x - (2y + 2x)}{y^2 + 2xy} = -\frac{2}{y}$$

depends only on y, we have as an integrating factor

$$\mu(y) = \exp\left(-\frac{2}{y}\right) = \frac{1}{y^2}.$$

Multiplying the ODE by y^{-2} gives us the exact equation

$$\left(1 + \frac{2x}{y}\right) - \frac{x^2}{y^2}y' = 0,$$

where $\hat{M}(x,y) = 1 + 2x/y$ and $\hat{N}(x,y) = -x^2$. There exists a function F such that $F_x = \hat{M}$ and $F_y = \hat{N}$, and the implicit solution to the ODE will be of the form $F(x,y) = c$. From $F_x = \hat{M}$ we obtain

$$F(x,y) = \int (1 + 2x/y)\,dx + g(y) = x + x^2/y + g(y),$$

and then from $F_y = \hat{N}$ we obtain

$$-x^2/y^2 + g'(y) = -x^2/y^2.$$

Hence $g'(y) = 0$, so that $g(y) = \hat{c}$ for some arbitrary constant $\hat{c}$. Therefore we have $F(x,y) = x + x^2/y + \hat{c}$, and so the solution is

$$x + \frac{x^2}{y} + \hat{c} = c.$$

The constant $\hat{c}$ can be absorbed by c to get simply $x + x^2y^{-1} = c$.

Another solution is $y \equiv 0$, which is not included in the one-parameter family of functions $x + x^2y^{-1} = c$.