Math 250 Exam #4 Key (Spring 2018)

1a The model is

$$\tfrac{1}{8}y'' + 2y' + 16y = 0, \quad y(0) = -\tfrac{3}{4}, \quad y'(0) = -2.$$

Auxiliary equation: $r^2/8 + 2r + 16 = 0$, or $r^2 + 16r + 128 = 0$. This has roots $-8 \pm 8i$, so

$$y(t) = e^{-8t}(c_1 \cos 8t + c_2 \sin 8t).$$

With $y(0) = -\frac{3}{4}$ we get $c_1 = -\frac{3}{4}$, and since

$$y'(t) = -8e^{-8t}[(c_1 - c_2)\cos 8t + (c_1 + c_2)\sin 8t],$$

with $y'(0) = -2$ we get $c_2 = -1$. So the equation of motion is

$$y(t) = -e^{-8t}\left(\tfrac{3}{4}\cos 8t + \sin 8t\right).$$

1b The object start out moving leftward, so maximum displacement to the left will occur at the smallest time $t_0 > 0$ for which $y'(t_0) = 0$. Now,

$$y'(t) = 0 \quad \Rightarrow \quad 2e^{-8t}(\cos 8t + 7\sin 8t) = 0 \quad \Rightarrow \quad \cos 8t = -7\sin 8t,$$

so $y'(t_0) = 0$ implies $\tan 8t_0 = -\frac{1}{7}$, and hence

$$8t_0 = \arctan\left(-\tfrac{1}{7}\right) + k\pi,$$

where we need k to be the smallest integer for which the right-hand side is positive. We find that $k = 1$, and so

$$t_0 = \tfrac{1}{8}\left[\arctan\left(-\tfrac{1}{7}\right) + \pi\right] = 2.999695... \approx 3 \text{ seconds}$$

2 Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0,$$

whence comes

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n + \sum_{n=0}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} 2c_n x^n = 0,$$

and then

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)c_{n+2} + nc_n - 2c_n\right]x^n = 0.$$

This implies that

$$(n+1)(n+2)c_{n+2} + nc_n - 2c_n = 0,$$

for all $n \geq 0$, and hence

$$c_{n+2} = \frac{2-n}{(n+1)(n+2)}c_n.$$

We now wind up our propeller beanies and calculate

$$c_2 = c_0, \quad c_3 = \frac{1}{3!}c_1, \quad c_4 = 0, \quad c_5 = \frac{-1}{5!}c_1, \quad c_6 = 0, \quad c_7 = \frac{3}{7!}c_1, \quad c_8 = 0,$$

$$c_9 = \frac{-3\cdot 5}{9!}c_1, \quad c_{10} = 0, \quad c_{11} = \frac{3\cdot 5\cdot 7}{11!}c_1,$$

and so on, giving

$$\begin{aligned}
y &= c_0 + c_1 x + c_0 x^2 + \frac{1}{3!}c_1 x^3 - \frac{1}{5!}c_1 x^5 + \frac{3}{7!}c_1 x^7 - \frac{3\cdot 5}{9!}c_1 x^9 + \frac{3\cdot 5\cdot 7}{11!}c_1 x^{11} + \cdots \\
&= c_0(1+x^2) + c_1\left(x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{3}{7!}x^7 - \frac{3\cdot 5}{9!}x^9 + \frac{3\cdot 5\cdot 7}{11!}x^{11} + \cdots\right) \\
&= c_0(1+x^2) + c_1\left(x + \frac{x^3}{6} + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]x^{2n+3}}{(2n+3)!}\right). \qquad (1)
\end{aligned}$$

This is the general solution to the ODE. Now we employ the initial condition $y(0) = 1$ to obtain $c_0 = 1$. Putting this into (1) and differentiating yields

$$y' = 2x + c_1\left(1 + \frac{x^2}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]x^{2n+21}}{(2n+2)!}\right).$$

Our initial condition $y'(0) = 0$ clearly implies $c_1 = 0$. Therefore the solution to the IVP is

$$y = 1 + x^2.$$

3 We have

$$\begin{aligned}
\mathcal{L}[f](s) &= \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^3 e^{(2-s)t}\,dt + \int_3^{\infty} e^{-st}\,dt \\
&= \frac{1}{2-s}\left[e^{(2-s)t}\right]_0^3 + \lim_{b\to\infty}\int_3^b e^{-st}\,dt = \frac{e^{3(2-s)}-1}{2-s} + \lim_{b\to\infty} -\frac{1}{s}\left[e^{-st}\right]_3^b \\
&= \frac{e^{6-3s}-1}{2-s} - \frac{1}{s}\lim_{b\to\infty}\left(e^{-sb} - e^{-3s}\right) = \frac{e^{6-3s}-1}{2-s} - \frac{1}{s}\left(0 - e^{-3s}\right)
\end{aligned}$$

if $s > 0$ and $s \neq 2$. Thus

$$\mathcal{L}[f](s) = \frac{e^{6-3s}-1}{2-s} + \frac{e^{-3s}}{s}, \qquad s \in (0,2)\cup(2,\infty).$$

4a Using the table provided,

$$\begin{aligned}
\mathcal{L}\big[t^5 - 7e^{-3t}\sin 4t\big](s) &= \mathcal{L}\big[t^5\big](s) - 7\mathcal{L}\big[e^{-3t}\sin 4t\big](s) \\
&= \frac{5!}{(s-0)^{5+1}} - 7\cdot\frac{4}{(s+3)^2+4^2} \\
&= \frac{120}{s^6} - \frac{28}{(s+3)^2+16}
\end{aligned}$$

4b Use a given trigonometric identity for this, along with the identity $\sin(-u) = -\sin(u)$ and the transform table:

$$\mathcal{L}[\sin t\cos 2t](s) = \mathcal{L}\big[\tfrac{1}{2}\sin(3t) + \tfrac{1}{2}\sin(-t)\big](s) = \tfrac{1}{2}\mathcal{L}[\sin(3t)](s) - \tfrac{1}{2}\mathcal{L}[\sin(t)](s)$$

$$= \frac{1}{2} \cdot \frac{3}{s^2 + 3^2} - \frac{1}{2} \cdot \frac{1}{s^2 + 1^2} = \frac{1}{2(s^2 + 9)} - \frac{1}{2(s^2 + 1)}.$$