Math 250 Exam #3 Key (Spring 2018)

1 Let $u = y'$, so ODE becomes $xu' = u + xu^2$, and thus

$$u' - \frac{1}{x}u = u^2.$$

This is a Bernoulli equation. Making the substitution $v = 1/u$ results in the equation $xv' + v = -x$, which is linear and becomes $(xv)' = -x$. Solving yields

$$v = -\frac{x}{2} + \frac{c}{x} = \frac{2c - x^2}{2x},$$

and hence, replacing arbitrary $2c$ with arbitrary c^2 (we are assuming $c > 0$ after all), we get

$$y' = u = \frac{2x}{2c - x^2} = \frac{2x}{c^2 - x^2} = \frac{1}{c - x} - \frac{1}{c + x}.$$

Integration yields

$$y = \ln|c - x| - \ln|c + x| + \hat{c} = \ln\left|\frac{c - x}{c + x}\right| + \hat{c}.$$

Note: assuming $c = 0$ or $c < 0$ would result in other kinds of solutions to the ODE.

2a Auxiliary equation is $r^2 + 3 = 0$, with solution $r = \pm i\sqrt{3}$. Particular solution will have form $y_p = (Ax^2 + Bx + C)e^{3x}$. We then find that

$$y_p'' + 3y_p = (12Ax^2 + 12Ax + 12Bx + 2A + 6B + 12C)e^{3x},$$

and so A, B, C must be such that

$$12Ax^2 + 12Ax + 12Bx + 2A + 6B + 12C = -48x^2.$$

This gives us the system

$$\begin{cases} 12A & = -48 \\ 12A + 12B & = 0 \\ 2A + 6B + 12C & = 0 \end{cases}$$

which has solution $A = -4$, $B = 4$, $C = -\frac{4}{3}$. A particular solution is therefore

$$y_p = \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}.$$

2b General solution is

$$y = \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x} + c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x.$$