Math 250 Exam #2 Key (Spring 2018)

1 First,

$$\frac{dI}{dt} = kI \quad \Rightarrow \quad \ln(I) = kt + c \quad \Rightarrow \quad I(t) = e^{kt+c} = Ce^{kt}.$$

We're given $I(0) = I_0$ and $I(1.5) = 0.07I_0$. With $T(0) = I_0$ we obtain $C = I_0$, so that $I(t) = I_0 e^{kt}$. Next,

$$I(1.5) = 0.07I_0 \quad \Rightarrow \quad 0.07I_0 = I_0 e^{1.5k} \quad \Rightarrow \quad e^{1.5k} = 0.07 \quad \Rightarrow \quad k = \frac{2}{3}\ln 0.07 \approx -1.773.$$

Thus

$$I(t) = I_0 e^{-1.773t},$$

and we find that $I(2.5) = I_0 e^{-1.773(2.5)} \approx 0.012I_0$. That is, 2.5 meters below the surface the light intensity is about 1.2% of the original intensity.

2 Newton's Law of Cooling states that $T'(t) = k[T(t) - M]$, where $M = 100$ is the water temperature. From this we obtain

$$\int \frac{1}{T - 100} dT = \int k\, dt \quad \Rightarrow \quad T(t) = 100 - Ce^{kt}.$$

We're given $T(0) = 20$ and $T(1) = 22$. With $T(0) = 20$ we find that $C = 80$, so

$$T(t) = 100 - 80e^{kt}.$$

With $T(1) = 22$ we get $22 = 100 - 80e^{k}$, which has solution $k = \ln\frac{39}{40} \approx -0.0253$. Hence

$$T(t) = 100 - 80e^{-0.0253t}.$$

To find the time when the temperature of the bar is 90°C, solve $90 = 100 - 80e^{-0.0253t}$ to get $t \approx 82.1$ seconds. To find when temperature is 98°C, solve $98 = 100 - 80e^{-0.0253t}$ to get $t \approx 145.8$ seconds.

3 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 4$. The volume of solution in the tank at time t is $V(t) = 400 + 3t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned}
x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\
&= \left(\frac{0.04 \text{ kg}}{1 \text{ L}}\right)\left(\frac{18 \text{ L}}{1 \text{ min}}\right) - \left(\frac{x(t) \text{ kg}}{V(t) \text{ L}}\right)\left(\frac{15 \text{ L}}{1 \text{ min}}\right) \\
&= 0.72 - \frac{15x(t)}{400 + 3t}.
\end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{15x}{3t + 400} = 0.72.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{15}{3t + 400}\, dt\right) = e^{5\ln(3t+400)} = (3t + 400)^5$$

to obtain

$$(3t+400)^5 x' + 15(3t+400)^4 x = 0.72(3t+400)^5,$$

which becomes

$$\left[(3t+400)^5 x\right]' = 0.72(3t+400)^5$$

and thus

$$(3t+400)^5 x = 0.72\int (3t+400)^5\,dt = 0.72\left[\frac{1}{18}(3t+400)^6\right] + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{3t+400}{25} + \frac{c}{(3t+400)^5}.$$

To find c we use the initial condition $x(0)=4$, giving $c = -12(400^5)$, and so

$$x(t) = \frac{3t+400}{25} - 12\left(\frac{400}{3t+400}\right)^5.$$

4 The check that each of the four functions is a solution to the ODE is routine. In the case of $\sin x$, for instance, we find that

$$(\sin x)^{(4)} + (\sin x)'' = \sin x - \sin x = 0,$$

as required. Now suppose there exist constants a, b, c, d such that

$$a + bx + c\cos x + d\sin x = 0$$

for all $x \in (-\infty, \infty)$. In particular, letting x equal 0, π, $\pi/2$ and $-\pi$, we find that

$$\begin{cases} a \phantom{{}+\pi b} + c \phantom{{}+d} = 0 \\ a + \pi b - c \phantom{{}+d} = 0 \\ a + \frac{\pi}{2}b \phantom{{}-c} + d = 0 \\ a - \pi b - c \phantom{{}+d} = 0 \end{cases}$$

This system is quickly found to have only the trivial solution $a = b = c = d = 0$, and therefore $\{1, x, \cos x, \sin x\}$ is a linearly independent set on $(-\infty, \infty)$. It follows that the set is a fundamental set of solutions to the ODE, and the general solution to the ODE is

$$y = c_1 + c_2 x + c_3\cos x + c_4 \sin x.$$

5 Put equation in standard form: $y'' + 2t^{-1}y' - 6t^{-2}y = 0$, so $P(t) = 2/t$. We're given that $y_1(t) = t^2$ is a solution. From this we obtain

$$y_2(t) = y_1(t)\int \frac{e^{-\int P(t)\,dt}}{y_1^2(t)}dt = t^2\int \frac{e^{-2\ln|t|}}{t^4}dt = t^2\int \frac{1}{t^6}dt = t^2\left(-\tfrac{1}{5}t^{-5} + c\right) = -\frac{1}{5t^3} + ct^2$$

for any $c \in \mathbb{R}$. If we let $c = 0$ then we get $y_2(t) = -1/5t^3$.

6a Auxiliary equation: $2r^2 - 7r + 3 = 0$, which has solutions $r = \frac{1}{2}, 3$. General solution:

$$y(x) = c_1 e^{x/2} + c_2 e^{3x}.$$

6b Auxiliary equation: $r^3 + 3r^2 - 4r - 12 = 0$. Now,

$$r^2(r+3) - 4(r+3) = 0 \quad \Rightarrow \quad (r+3)(r^2-4) = 0 \quad \Rightarrow \quad (r+3)(r-2)(r+2) = 0,$$

so solutions are $r = -3, 2, -2$. General solution:

$$y(x) = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{-2x}.$$

7 Auxiliary equation is $r^2 - 2r + 1 = 0$, which has double root 1. General solution to ODE is thus

$$y = c_1 e^x + c_2 x e^x.$$

With $y(0) = 5$ we immediately find that $c_1 = 5$. With

$$y' = c_1 e^x + c_2 (x+1) e^x$$

and $y'(0) = 10$ we find that $c_2 = 5$. Solution to IVP is thus

$$y = 5e^x + 5xe^x.$$