# Math 250 Exam #1 Key (Spring 2018)

**1** We have

$$y' = \frac{xy}{\sqrt{6-y}}, \quad y(x_0) = y_0,$$

which is an IVP that will have a unique solution if

$$f(x,y) = \frac{xy}{\sqrt{6-y}} \quad \text{and} \quad f_y(x,y) = \frac{12x - xy}{2(6-y)^{3/2}}$$

are both continuous on an open rectangle containing $(x_0, y_0)$. This is so for $(x_0, y_0) \in \mathbb{R}^2$ such that $6 - y_0 > 0$, or equivalently $y_0 \in (-\infty, 6)$. That is, the IVP will have a unique solution if

$$(x_0, y_0) \in \big\{(x,y) : y < 6\big\}.$$

**2** The force of air resistance on the body is $kv^2$ for some constant $k$, while the force of gravity on the body is $mg$. The sum of these forces equals $mv'$, where $v'$ is the acceleration of the body. Thus: $kv^2 + mg = mv'$.

**3** Apply separation of variables to obtain

$$\int \frac{1}{N} dN = \int (te^{t+2} - 1)\, dt \quad \Rightarrow \quad \ln|N| = (t-1)e^{t+2} - t + c \quad \Rightarrow \quad |N| = Ce^{(t-1)e^{t+2} - t},$$

where $C = e^c > 0$. Thus

$$N = Ce^{(t-1)e^{t+2} - t}$$

for $C \neq 0$. However, it can be seen that $N \equiv 0$ is also a solution to the ODE, and so we conclude that

$$N = Ce^{(t-1)e^{t+2} - t}$$

for $C \in \mathbb{R}$ is a one-parameter family of solutions.

**4a** Separation of variables gives

$$\int 4y\, dy = \int (3x - 1)\, dx \quad \Rightarrow \quad 2y^2 = \frac{3}{2}x^2 - x + c.$$

With $y(-2) = -1$ we obtain $c = -6$, and so we have

$$y^2 = \tfrac{3}{4}x^2 - \tfrac{1}{2}x - 3 \quad \text{or} \quad |y| = \sqrt{\tfrac{3}{4}x^2 - \tfrac{1}{2}x - 3}.$$

This is good enough for us. However, since $y < 0$ at the initial point $(-2, -1)$, we can resolve the absolute value:

$$y = -\sqrt{\tfrac{3}{4}x^2 - \tfrac{1}{2}x - 3}.$$

**4b** We must have $\frac{3}{4}x^2 - \frac{1}{2}x - 3 > 0$, or equivalently $x \in \big(-\infty, \frac{1-\sqrt{37}}{3}\big) \cup \big(\frac{1+\sqrt{37}}{3}, \infty\big)$. But since $x < 0$ at the initial point, it follows that the interval of validity is $\big(-\infty, \frac{1-\sqrt{37}}{3}\big)$.

**5** Standard form is $y' + y/x = (x^2+1)/x$. Integrating factor is

$$\mu(x) = e^{\int (1/x) dx} = e^{\ln x} = x,$$

which we multiply the ODE by to get

$$xy' + y = x^2 + 1,$$

or

$$(xy)' = x^2 + 1.$$

Integrate both sides:

$$xy = \frac{1}{3}x^3 + x + c.$$

Therefore

$$y(x) = \frac{x^2}{3} + \frac{c}{x} + 1.$$

**6** We find a function $F$ such that $F_x(x,y) = e^x + y$ and $F_y(x,y) = 2 + x + ye^y$. Now,

$$F(x,y) = \int (e^x + y) dx = e^x + xy + g(y)$$

for arbitrary differentiable function $g$. Then

$$2 + x + ye^y = F_y(x,y) = x + g'(y) \quad \Rightarrow \quad g'(y) = 2 + ye^y \quad \Rightarrow \quad g(y) = 2y + ye^y - e^y,$$

so

$$F(x,y) = e^x + xy + 2y + ye^y - e^y.$$

Solution to ODE is $F(x,y) = c$; that is,

$$e^x + xy + 2y + ye^y - e^y = c.$$

Initial condition gives $y = 1$ when $x = 0$, so $1 + 0 + 2 + e - e = c$, or $c = 3$, and therefore the solution to the IVP is

$$e^x + (y-1)e^y + (x+2)y = 3.$$

**7** Rewrite the equation as

$$y' = \frac{1 + (y/x)e^{y/x}}{e^{y/x}}.$$

Let $u = y/x$, so $y' = xu' + u$. Equation becomes

$$xu' + u = \frac{1 + ue^u}{e^u} \quad \Rightarrow \quad u' = \frac{1}{xe^u} \quad \Rightarrow \quad \int \frac{1}{x}\, dx = \int e^u\, du \quad \Rightarrow \quad \ln|x| = e^u + c$$

$$\Rightarrow \quad |x| = Ce^{e^u} = Ce^{e^{y/x}}, \quad C > 0.$$

Therefore

$$x = Ce^{e^{y/x}}, \quad C \neq 0.$$

**8** Rewrite equation thus: $y' + (6/x)y = 3y^{4/3}$. This is Bernoulli with $n = 4/3$, $P(x) = 6/x$, and $Q(x) = 3$. Letting $v = y^{1-n} = y^{-1/3}$, we obtain the linear equation

$$v' - \frac{2}{x}v = -1.$$

Multiplying by the integrating factor $\mu(x) = x^{-2}$ yields

$$\frac{1}{x^2}v' - \frac{2}{x^3}v = -\frac{2}{x^2} \quad \Rightarrow \quad \left(\frac{1}{x^2}v\right)' = -\frac{1}{x^2} \quad \Rightarrow \quad \frac{v}{x^2} = \frac{1}{x} + c \quad \Rightarrow \quad v = x + cx^2,$$

whence

$$y = \frac{1}{(x + cx^2)^3}.$$

Also $y \equiv 0$ is a solution.