Math 250 Exam #4 Key (Spring 2017)

1 This is a nonlinear equation with y missing. Let $u = y'$, so $u' = y''$ and the ODE becomes $x^2u' + u^2 = 0$. This is separable, yields

$$\int \frac{1}{u^2}\,du = -\int \frac{1}{x^2}\,dx \quad\Rightarrow\quad y' = u = -\frac{x}{1+\alpha x}$$

for arbitrary α. If $\alpha \neq 0$,

$$y = \int \frac{x}{\alpha x - 1}\,dx = \frac{1}{\alpha}\int\left(1 + \frac{1}{\alpha x - 1}\right)dx = \frac{x}{\alpha} + \frac{1}{\alpha^2}\ln|\alpha x - 1| + \beta$$

for arbitrary β. If $\alpha = 0$,

$$y' = -x \quad\Rightarrow\quad y = -\frac{1}{2}x^2 + \beta.$$

Therefore the solution set of the ODE contains the family of functions

$$y = \begin{cases} \dfrac{\alpha x + \ln|\alpha x - 1|}{\alpha^2} + \beta, & \alpha \neq 0,\ \beta \in \mathbb{R} \\[2ex] -\dfrac{x^2}{2} + \beta, & \beta \in \mathbb{R}. \end{cases}$$

2 The equation is nonlinear with x missing, so we cast y in the role of the independent variable and make the substitution $u = y' = dy/dx$. Then

$$y'' = u' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy},$$

and the ODE becomes

$$\frac{u}{(1+u^2)^{3/2}}\frac{du}{dy} = \kappa.$$

The equation is separable, giving

$$\int \frac{u}{(1+u^2)^{3/2}}\,du = \int \kappa\,dy \quad\Rightarrow\quad -\frac{1}{\sqrt{1+u^2}} = \kappa y - \beta$$

where β is an arbitrary constant. Letting $\beta = 0$, we get

$$\frac{1}{1+(y')^2} = (\kappa y)^2 \quad\Rightarrow\quad y' = \pm\sqrt{\frac{1-(\kappa y)^2}{(\kappa y)^2}} = \pm\frac{\sqrt{1-(\kappa y)^2}}{\kappa y}.$$

This is again a separable equation, becoming

$$\pm\int \frac{\kappa y}{\sqrt{1-(\kappa y)^2}}\,dy = \int dx.$$

Making the substitution $w = 1 - (\kappa y)^2$ yields

$$\pm\int \frac{-1}{2\kappa\sqrt{w}}\,dw = x - \alpha$$

for arbitrary α. Let $\alpha = 0$, so that

$$\pm\frac{\sqrt{1-(\kappa y)^2}}{\kappa} = x.$$

With the $\pm$ we have the right and left halves of a circle. Squaring and doing some algebra gives

$$x^2 + y^2 = \frac{1}{\kappa^2},$$

a circle with center $(0,0)$ and radius $1/\kappa$. Setting $\kappa \equiv 1$ yields $y^2 = 1 - x^2$. One choice for the function f is therefore $f(x) = \sqrt{1-x^2}$.

3 From Hooke's Law we have $10 = k(2)$, and so $k = 5$ lb/ft is the spring constant. The mass m is given by $W = mg$, so $m = W/g = 10/32 = 5/16$ slug. The equation of motion is thus

$$\frac{5}{16}y'' + \beta y' + 5y = 0 \quad\Rightarrow\quad 5y'' + 16\beta y' + 80y = 0.$$

The auxiliary equation is

$$5r^2 + 16\beta r + 80 = 0. \tag{1}$$

The mass-spring system is overdamped if (1) has distinct real roots, critically damped if (1) has a double root, and underdamped if (1) has complex conjugate roots. By the quadratic equation we have

$$r = \frac{-16\beta \pm \sqrt{(16\beta)^2 - 4(5)(80)}}{2(5)} = \frac{-8\beta \pm 4\sqrt{4\beta^2 - 25}}{5},$$

and so the system is overdamped if $4\beta^2 - 25 > 0$, which gives $\beta > \frac{5}{2}$. Critical damping occurs if $\beta = \frac{5}{2}$, and underdamping occurs if $0 < \beta < \frac{5}{2}$.

4 Substituting

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

into the ODE gives

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = x^2 \sum_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad c_1 + 2c_2 x + \sum_{n=3}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} c_n x^{n+2},$$

and then

$$c_1 + 2c_2 x + \sum_{n=2}^{\infty} (n+1) c_{n+1} x^n = \sum_{n=2}^{\infty} c_{n-2} x^n.$$

Now we have

$$c_1 + 2c_2 x + \sum_{n=2}^{\infty} \big[(n+1)c_{n+1} - c_{n-2}\big] x^n = 0$$

for all x in some open interval, implying that $c_1 = c_2 = 0$, and $(n+1)c_{n+1} - c_{n-2} = 0$ for all $n \geq 2$. This leaves c_0 to be arbitrary, and

$$c_3 = \frac{c_0}{3}, \quad c_4 = 0, \quad c_5 = 0, \quad c_6 = \frac{c_3}{6} = \frac{c_0}{3\cdot 6}, \quad c_7 = 0, \quad c_8 = 0, \quad c_9 = \frac{c_6}{9} = \frac{c_0}{3\cdot 6\cdot 9},$$

and so on. Thus

$$y = c_0 + \frac{c_0}{3}x^3 + \frac{c_0}{3\cdot 6}x^6 + \frac{c_0}{3\cdot 6\cdot 9}x^9 + \frac{c_0}{3\cdot 6\cdot 9\cdot 12}x^{12} + \cdots = \sum_{n=0}^{\infty} \frac{c_0 x^{3n}}{3^n\, n!},$$

or equivalently

$$y = c \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^3}{3} \right)^n .$$

for arbitrary $c \in \mathbb{R}$. (Solving the ODE by separation of variables gives $y = ce^{x^3/3}$, which is the same thing.)

5 Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^n = 0,$$

whence comes

$$\sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2} x^n + \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 2c_n x^n = 0,$$

and then

$$\sum_{n=0}^{\infty} \big[(n+1)(n+2)c_{n+2} + nc_n - 2c_n \big] x^n = 0.$$

This implies that

$$(n+1)(n+2)c_{n+2} + nc_n - 2c_n = 0,$$

for all $n \geq 0$, and hence

$$c_{n+2} = \frac{2-n}{(n+1)(n+2)} c_n.$$

We now wind up our propeller beanies and calculate

$$c_2 = c_0, \quad c_3 = \frac{1}{3!}c_1, \quad c_4 = 0, \quad c_5 = \frac{-1}{5!}c_1, \quad c_6 = 0, \quad c_7 = \frac{3}{7!}c_1, \quad c_8 = 0,$$
$$c_9 = \frac{-3 \cdot 5}{9!}c_1, \quad c_{10} = 0, \quad c_{11} = \frac{3 \cdot 5 \cdot 7}{11!}c_1,$$

and so on, giving

$$\begin{aligned} y &= c_0 + c_1 x + c_0 x^2 + \frac{1}{3!}c_1 x^3 - \frac{1}{5!}c_1 x^5 + \frac{3}{7!}c_1 x^7 - \frac{3 \cdot 5}{9!}c_1 x^9 + \frac{3 \cdot 5 \cdot 7}{11!}c_1 x^{11} + \cdots \\ &= c_0(1+x^2) + c_1 \left(x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{3}{7!}x^7 - \frac{3 \cdot 5}{9!}x^9 + \frac{3 \cdot 5 \cdot 7}{11!}x^{11} + \cdots \right) \\ &= c_0(1+x^2) + c_1 \left(x + \frac{x^3}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n [1 \cdot 3 \cdot 5 \cdots (2n-1)] x^{2n+3}}{(2n+3)!} \right). \end{aligned} \tag{2}$$

This is the general solution to the ODE. Now we employ the initial condition $y(0) = 1$ to obtain $c_0 = 1$. Putting this into (2) and differentiating yields

$$y' = 2x + c_1 \left(1 + \frac{x^2}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n [1 \cdot 3 \cdot 5 \cdots (2n-1)] x^{2n+2}}{(2n+2)!} \right).$$

Our initial condition $y'(0) = 0$ clearly implies $c_1 = 0$. Therefore the solution to the IVP is

$$y = 1 + x^2.$$