# Math 250 Exam #3 Key (Spring 2017)

**1** Standard form is $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$. A second particular solution:

$$y_2(x) = x^4 \int \frac{e^{-\int(-7/x)\,dx}}{(x^4)^2} dx = x^4 \int \frac{e^{7\ln x}}{x^8}\,dx = x^4 \int \frac{1}{x}\,dx = x^4 \ln x.$$

**2a** Auxiliary equation is $r^3 - 3r^2 + 3r - 1 = 0$ has root 1 with multiplicity 3, and since the nonhomogeneity has form $P_m(x)e^{\alpha x}$ with $\alpha = 1$ (and $m = 0$), we have $s = 3$ in the form for the particular solution given by the Method of Undetermined Coefficients:

$$y_p(x) = x^s e^{\alpha x} \sum_{k=0}^{m} A_k x^k = Ax^3 e^x \tag{1}$$

Now,

$$y_p'(x) = (3Ax^2 + Ax^3)e^x,$$
$$y_p''(x) = (6Ax + 6Ax^2 + Ax^3)e^x,$$
$$y_p'''(x) = (6A + 18Ax + 9Ax^2 + Ax^3)e^x,$$

which when put into the ODE results in $6A = -4$, and thus $A = -\frac{2}{3}$. Therefore $y_p(x) = -\frac{2}{3}x^3e^x$.

**2b** Given that 1 is a root of the auxiliary equation with multiplicity 3,

$$y(x) = -\frac{2}{3}x^3e^x + c_1e^x + c_2xe^x + c_3x^2e^x$$

is the general solution.

**3a** Nonhomogeneity $f(x) = x^2 - 2x$ has form $P_m(x)e^{\alpha x}$ with $m = 2$ and $\alpha = 0$. The auxiliary equation is

$$\frac{1}{4}r^2 + r + 1 = 0,$$

which has root $-2$ with multiplicity 2. Since $\alpha = 0$ is not a root, we have $s = 0$ in the form (1) for the particular solution, so that $y_p(x) = A_1 + A_2x + A_3x^2$. Rewrite this as $y_p(x) = A + Bx + Cx^2$. Then

$$y_p'(x) = B + 2Cx \quad \text{and} \quad y_p''(x) = 2C.$$

Putting these results into the ODE (multiplied by 4) gives

$$4Cx^2 + (4B + 8C)x + (4A + 4B + 2C) = 4x^2 - 8x,$$

which leads to the system

$$\begin{cases} 4C = 4 \\ 4B + 8C = -8 \\ 4A + 4B + 2C = 0 \end{cases}$$

The solution is $A = \frac{7}{2}$, $B = -4$, $C = 1$. Particular solution is therefore

$$y_p(x) = x^2 - 4x + \frac{7}{2}$$

**3b** With $-2$ a double root of the auxiliary equation, the general solution is

$$y(x) = x^2 - 4x + \frac{7}{2} + c_1 e^{-2x} + c_2 x e^{-2x}.$$

**4** Auxiliary equation $r^2+3r+2 = 0$ has roots $-2$ and $-1$, and so $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$ form a fundamental solution set for $y'' + 3y' + 2y = 0$. Now,

$$\mathcal{W}[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = -e^{-3x},$$

and so, making the substitution $w = e^x$, we find that

$$u_1(x) = \int \frac{-e^{-2x}\sin(e^x)}{-e^{-3x}}\,dx = \int e^x \sin(e^x)\,dx = \int \sin w\,dw = -\cos w = -\cos(e^x)$$

and (using integration by parts)

$$\begin{aligned} u_2(x) &= \int \frac{-e^{-x}\sin(e^x)}{-e^{-3x}}\,dx = -\int e^{2x}\sin(e^x)\,dx = -\int w \sin w\,dw = w\cos w - \sin w \\ &= e^x\cos(e^x) - \sin(e^x). \end{aligned}$$

Therefore

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = -e^{-x}\cos(e^x) + [e^x\cos(e^x) - \sin(e^x)]e^{-2x} = -e^{-2x}\sin(e^x).$$

General solution:

$$y(x) = -e^{-2x}\sin(e^x) + c_1 e^{-x} + c_2 e^{-2x}.$$