Math 250 Exam #2 Key (Spring 2017)

1a Basic model for radioactive decay: $x(t) = x_0 e^{-kt}$. Given: $x(0) = 100$. Thus $x_0 = 100$ and we have $x(t) = 100e^{-kt}$. Find k. Given: $x(6) = 96.6$. So $96.6 = 100e^{-kt}$, which solves to give $k \approx 0.00577$. Therefore

$$x(t) = 100e^{-0.00577t}.$$

After 24 hours the quantity of isotope remaining is

$$x(24) = 100e^{-0.00577(24)} \approx 87.1 \text{ mg}.$$

1b Find t for which $x(t) = 10$:

$$100e^{-0.00577t} = 10 \quad\Rightarrow\quad e^{-0.00577t} = 0.1 \quad\Rightarrow\quad t = -\frac{\ln 0.1}{0.00577} \approx 399.1 \text{ hr}.$$

2 Newton's Law of Cooling states that $T'(t) = k[T(t) - M]$, where $M = 68$ is the temperature of the house. From this ODE we obtain

$$\int \frac{1}{T-68}\,dT = \int k\,dt \quad\Rightarrow\quad T(t) = 68 + Ce^{kt}.$$

Letting $t = 0$ be the time of death and $t = \tau$ the time of discovery, we're given $T(0) = 98.6$, $T(\tau) = 83$, $T(\tau + 1) = 77$. With $T(0) = 98.6$ we find that $C = 30.6$. So

$$T(t) = 68 + 30.6e^{kt}.$$

Now, with $T(\tau) = 83$ and $T(\tau + 1) = 77$ we obtain

$$83 = 68 + 30.6e^{k\tau} \quad\Rightarrow\quad e^{k\tau} = \frac{25}{51} \tag{1}$$

and

$$77 = 68 + 30.6e^{k(\tau+1)} = 68 + 30.6e^{k\tau}e^{k}, \tag{2}$$

respectively. Substituting (1) into (2) gives

$$77 = 68 + 30.6\left(\frac{25}{51}e^{k}\right),$$

which solves to give $k = \ln 0.6$. Putting this into (1) yields

$$e^{\tau \ln 0.6} = \frac{25}{51} \quad\Rightarrow\quad \tau \approx 1.40 \text{ hr}.$$

Therefore 1.40 hours elapsed between the time of death and the time the body was found.

3 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 4$. The volume of solution in the tank at time t is $V(t) = 400 + 3t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned} x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\ &= \left(\frac{0.04 \text{ kg}}{1 \text{ L}}\right)\left(\frac{18 \text{ L}}{1 \text{ min}}\right) - \left(\frac{x(t) \text{ kg}}{V(t) \text{ L}}\right)\left(\frac{15 \text{ L}}{1 \text{ min}}\right) \end{aligned}$$

$$= 0.72 - \frac{15x(t)}{400+3t}.$$

Thus we have a linear first-order ODE:

$$x' + \frac{15x}{3t+400} = 0.72.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{15}{3t+400}\,dt\right) = e^{5\ln(3t+400)} = (3t+400)^5$$

to obtain

$$(3t+400)^5 x' + 15(3t+400)^4 x = 0.72(3t+400)^5,$$

which becomes

$$\left[(3t+400)^5 x\right]' = 0.72(3t+400)^5$$

and thus

$$(3t+400)^5 x = 0.72\int (3t+400)^5\,dt = 0.72\left[\frac{1}{18}(3t+400)^6\right] + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{3t+400}{25} + \frac{c}{(3t+400)^5}.$$

To find c we use the initial condition $x(0) = 4$, giving $c = -12(400^5)$, and so

$$x(t) = \frac{3t+400}{25} - 12\left(\frac{400}{3t+400}\right)^5.$$

4 Let $y_1 = x^3$ and $y_2 = x^4$. We have $y_1' = 3x^2$ and $y_1'' = 6x$. Substituting into the ODE gives

$$x^2(6x) - 6x(3x^2) + 12x^3 = 0 \quad\Leftrightarrow\quad 6x^3 - 18x^3 + 12x^3 = 0 \quad\Leftrightarrow\quad 0 = 0.$$

Thus we see that y_1 is a solution to the ODE. A similar procedure will show that y_2 is a solution as well.

Suppose $c_1x^3 + c_2x^4 = 0$ for all $x \in (0,\infty)$. When $x = 1$ we get $c_1 + c_2 = 0$, and when $x = 2$ we get $8c_1 + 16c_2 = 0$. To satisfy both of these equations requires that $c_1 = c_2 = 0$. Therefore $\{x^3, x^4\}$ is a linearly independent set of functions on $(0,\infty)$.

By definition a fundamental set of solutions to a 2nd-order ODE on an interval I is a set $\{y_1, y_2\}$ of two functions that is linearly independent on I, with y_1 and y_2 each satisfying the ODE on I, so we're done.

5a Auxiliary equation: $2r^2 - 7r + 3 = 0$, which has solutions $r = \frac{1}{2}, 3$. General solution:

$$y(x) = c_1e^{x/2} + c_2e^{3x}.$$

5b Auxiliary equation: $r^3 + 3r^2 - 4r - 12 = 0$. Now,

$$r^2(r+3) - 4(r+3) = 0 \quad\Rightarrow\quad (r+3)(r^2-4) = 0 \quad\Rightarrow\quad (r+3)(r-2)(r+2) = 0,$$

so solutions are $r = -3, 2, -2$. General solution:

$$y(x) = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{-2x}.$$

6 Auxiliary equation: $r^2 - 6r + 25 = 0$, which has solutions

$$r = \frac{6 \pm \sqrt{6^2 - 4(25)}}{2} = 3 \pm 4i.$$

General solution to ODE:

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x),$$

and hence

$$y'(x) = 3e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x).$$

with the initial conditions given we find that $c_1 = 3$ and $c_2 = -2$. Therefore the solution to the IVP is

$$y(x) = e^{3x}(3\cos 4x - 2\sin 4x).$$

7 Roots to the auxiliary equation must be 0 and $2 \pm 5i$. Auxiliary equation:

$$r\big[r - (2+5i)\big]\big[r - (2-5i)\big] = 0 \quad \Rightarrow \quad r^3 - 4r^2 + 29r = 0.$$

The ODE with this auxiliary equation:

$$y''' - 4y'' + 29y' = 0.$$