Math 250 Exam #4 Key (Spring 2016)

1 The IVT is

$$y'' + 10y' + 16y = 0, \quad y(0) = 1, \quad y'(0) = -12.$$

The auxiliary equation $r^2 + 10r + 16 = 0$ has roots -8 and -2, and so the general solution to the ODE is

$$y(t) = c_1 e^{-2t} + c_2 e^{-8t}.$$

With the initial conditions we find that $c_1 = -\frac{2}{3}$ and $c_2 = \frac{5}{3}$. The equation of motion is therefore

$$y(t) = -\tfrac{2}{3}e^{-2t} + \tfrac{5}{3}e^{-8t}.$$

2 We have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 5\sum_{n=0}^{\infty} c_n x^{n+2} &= \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - 5\sum_{n=2}^{\infty} c_{n-2}x^n \\ &= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=2}^{\infty} 5c_{n-2}x^n \\ &= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}\big[(n+1)(n+2)c_{n+2} - 5c_{n-2}\big]x^n.\end{aligned}$$

3 Substituting

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

into the ODE gives

$$\sum_{n=1}^{\infty} nc_n x^{n-1} = x\sum_{n=0}^{\infty} c_n x^n,$$

so

$$c_1 + \sum_{n=2}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty} c_n x^{n+1},$$

and then with reindexing we obtain

$$c_1 + \sum_{n=1}^{\infty}(n+1)c_{n+1}x^n = \sum_{n=1}^{\infty} c_{n-1}x^n \quad \Rightarrow \quad c_1 + \sum_{n=1}^{\infty}\big[(n+1)c_{n+1} - c_{n-1}\big]x^n = 0.$$

This implies that $c_1 = 0$, and $(n+1)c_{n+1} - c_{n-1} = 0$ for all $n \geq 1$. This leaves c_0 to be arbitrary, and

$$c_2 = \frac{c_0}{2}, \quad c_3 = \frac{c_1}{3} = 0, \quad c_4 = \frac{c_2}{4} = \frac{c_0}{2\cdot 4}, \quad c_5 = 0, \quad c_6 = \frac{c_0}{2\cdot 4\cdot 6}, \quad c_7 = 0, \quad c_8 = \frac{c_0}{2\cdot 4\cdot 6\cdot 8},$$

and so on. Thus

$$y = c_0 + \frac{c_0}{2}x^2 + \frac{c_0}{2\cdot 4}x^4 + \frac{c_0}{2\cdot 4\cdot 6}x^6 + \frac{c_0}{2\cdot 4\cdot 6\cdot 8}x^8 + \cdots = c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{n!\cdot 2^n},$$

or equivalently

$$y = c\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^2}{2}\right)^n.$$

for arbitrary $c \in \mathbb{R}$. (Solving the ODE by separation of variables gives $y = ce^{x^2/2}$, which is the same thing.)

4 Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + 8\sum_{n=0}^{\infty} c_n x^n = 0,$$

whence comes

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=0}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} 8c_n x^n = 0,$$

and then

$$\sum_{n=0}^{\infty}\big[(n+1)(n+2)c_{n+2} - 2nc_n + 8c_n\big]x^n = 0.$$

This implies that

$$(n+1)(n+2)c_{n+2} - 2nc_n + 8c_n = 0,$$

for all $n \geq 0$, and hence

$$c_{n+2} = \frac{2n-8}{(n+1)(n+2)}c_n.$$

We now calculate

$$c_2 = -4c_0, \quad c_3 = \frac{-6}{3!}c_1, \quad c_4 = \frac{4}{3}c_0, \quad c_5 = \frac{(-6)(-2)}{5!}c_1, \quad c_6 = 0, \quad c_7 = \frac{(-6)(-2)(2)}{7!}c_1,$$

$$c_8 = 0, \quad c_9 = \frac{(-6)(-2)(2)(6)}{9!}c_1, \quad c_{10} = 0,$$

and in general

$$c_{2n+1} = \frac{(-6)(-2)(2)\cdots(4n-10)}{(2n+1)!}c_1$$

for $n \geq 0$, and $c_{2n} = 0$ for $n \geq 3$. Now, since

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n}x^{2n} + \sum_{n=0}^{\infty} c_{2n+1}x^{2n+1},$$

we conclude that

$$y = c_0\left(1 - 4x^2 + \frac{4}{3}x^4\right) + c_1\left(x + \sum_{n=1}^{\infty}\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n+1)!}x^{2n+1}\right).$$

This along with the initial condition $y(0) = 3$ yields $c_0 = 3$. From

$$y' = -8c_0 x + \frac{16}{3}c_0 x^3 + c_1\left(1 + \sum_{n=1}^{\infty}\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n)!}x^{2n}\right)$$

and the initial condition $y'(0) = 0$ we get $c_1 = 0$. Therefore

$$y = 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 3 - 12x^2 + 4x^4$$

is the solution to the IVP.

5 We have

$$\mathcal{L}[f](s) = \int_0^8 e^{-st}dt + \int_8^\infty te^{-st}dt = -\frac{1}{s}(e^{-8s} - 1) + \left(\frac{8}{s} + \frac{1}{s^2}\right)e^{-8s} = \frac{1}{s} + \left(\frac{7}{s} + \frac{1}{s^2}\right)e^{-8s}.$$

6a $\mathcal{L}[-2t^5](s) = -2\mathcal{L}[t^5](s) = -2 \cdot \dfrac{5!}{s^6} = -\dfrac{240}{s^6}.$

6b $\mathcal{L}[(2t-1)^3](s) = \mathcal{L}[8t^3 - 12t^2 + 6t - 1](s) = 8 \cdot \dfrac{3!}{s^4} - 12 \cdot \dfrac{2!}{s^3} + 6 \cdot \dfrac{1!}{s^2} - \dfrac{1}{s} = \dfrac{48}{s^4} - \dfrac{24}{s^3} + \dfrac{6}{s^2} - \dfrac{1}{s}.$

6c We have

$$e^t \sinh t = e^t\left(\frac{e^t - e^{-t}}{2}\right) = \frac{1}{2}e^{2t} - \frac{1}{2},$$

and so

$$\mathcal{L}\left[e^t \sinh t\right](s) = \frac{1}{2}\mathcal{L}\left[e^{2t}\right](s) - \frac{1}{2}\mathcal{L}[1](s) = \frac{1}{2} \cdot \frac{1}{s-2} - \frac{1}{2} \cdot \frac{1}{s} = \frac{1}{2(s-2)} - \frac{1}{2s}.$$