Math 250 Exam #3 Key (Spring 2016)

1 Put equation in standard form: $y'' + 2t^{-1}y' - 6t^{-2}y = 0$, so $P(t) = 2/t$. We're given that $y_1(t) = t^2$ is a solution. From this we obtain

$$y_2(t) = y_1(t)\int \frac{e^{-\int P(t)\,dt}}{y_1^2(t)}dt = t^2\int \frac{e^{-2\ln|t|}}{t^4}dt = t^2\int \frac{1}{t^6}dt = t^2\left(-\tfrac{1}{5}t^{-5} + c\right) = -\frac{1}{5t^3} + ct^2$$

for any $c \in \mathbb{R}$. If we let $c = 0$ then we get $y_2(t) = -1/5t^3$.

2a First consider $y'' + 2y' = 2t + 5$. Auxiliary equation is $r^2 + 2r = 0$, which has roots $r = -2, 0$. Now, the nonhomogeneity $f_1(t) = 2t + 5$ has the form $P_m(t)e^{\alpha t}$ with $m = 1$ and $\alpha = 0$, and since 0 is a root of the auxiliary equation we will need $s = 1$ in the form for the particular solution y_{p_1}. We have:

$$y_{p_1}(t) = t^s e^{\alpha t}\sum_{k=0}^{m} A_k t^k = t(A_0 + A_1 t) = At + Bt^2$$

(it's convenient to let $A = A_0$ and $B = A_1$). Thus $y_{p_1}'(t) = A + 2Bt$ and $y_{p_1}''(t) = 2B$. Putting all this into $y'' + 2y' = 2t + 5$ gives

$$2B + 2(A + 2Bt) = 2t + 5 \quad\Rightarrow\quad 4Bt + (2A + 2B) = 2t + 5,$$

so that $4B = 2$ and $2A + 2B = 5$, and finally $A = 2$ and $B = \frac{1}{2}$. Therefore $y_{p_1}(t) = 2t + \frac{1}{2}t^2$.

Next consider $y'' + 2y' = -e^{-2t}$. The nonhomogeneity $f_2(t) = -e^{-2t}$ has the form $P_m(t)e^{\alpha t}$ with $m = 0$ and $\alpha = -2$, and since -2 is a root of the auxiliary equation we will need $s = 1$ in the form for the particular solution y_{p_2}. We have

$$y_{p_2}(t) = t^s e^{\alpha t}\sum_{k=0}^{m} A_k t^k = tAe^{-2t},$$

so

$$y_{p_2}'(t) = (-2At + A)e^{-2t} \quad\text{and}\quad y_{p_2}''(t) = (4At - 4A)e^{-2t}.$$

Putting all this into $y'' + 2y' = -e^{-2t}$ and simplifying gives $-2Ae^{-2t} = -e^{-2t}$, and thus $A = \frac{1}{2}$. Therefore $y_{p_2}(t) = \frac{1}{2}te^{-2t}$.

By the Superposition Principle we conclude that

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) = 2t + \tfrac{1}{2}t^2 + \tfrac{1}{2}te^{-2t}$$

is a particular solution of the original equation.

2b General solution is

$$y(t) = c_1 + c_2e^{-2t} + 2t + \tfrac{1}{2}t^2 + \tfrac{1}{2}te^{-2t}.$$

3 The nonhomogeneity is $f(t) = -2$, so has form $P_m(t)e^{\alpha t}$ with $m = 0$ and $\alpha = 0$. Auxiliary equation: $r^2 + 4 = 0$, which has roots $r = \pm 2i$. Thus $\alpha = 0$ is not a root of the auxiliary equation. By the Method of Undetermined Coefficients we have $y_p(t) = A$, which when put into the ODE easily gives $A = -\frac{1}{2}$, and so $y_p(t) = -\frac{1}{2}$. General solution is therefore

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \tfrac{1}{2}.$$

Now, from $y(\pi/8) = \frac{1}{2}$ we obtain $c_1 + c_2 = \sqrt{2}$, and from

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t.$$

and $y'(\pi/8) = 2$ we obtain $-c_1 + c_2 = \sqrt{2}$. Adding $c_1 + c_2 = \sqrt{2}$ and $-c_1 + c_2 = \sqrt{2}$ gives $2c_2 = 2\sqrt{2}$, or $c_2 = \sqrt{2}$, from which it follows that $c_1 = 0$. Therefore

$$y(t) = \sqrt{2} \sin 2t - \tfrac{1}{2}$$

is the solution to the IVP.

4 From the auxiliary equation $2r^2 + 2r + 1 = 0$ we obtain $r = -\frac{1}{2} \pm \frac{1}{2}i$, and so

$$y_1(t) = e^{-t/2} \cos(t/2) \quad \text{and} \quad y_2(t) = e^{-t/2} \sin(t/2)$$

are two linearly independent solutions to $2y'' + 2y' + y = 0$. Once we get $y_1'(t)$ and $y_2'(t)$, we can find the Wronskian of y_1 and y_2:

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \tfrac{1}{2}e^{-t}.$$

Now, with $a_2 = 2$ and $f(t) = 4\sqrt{t}$, we have

$$v_1(t) = \frac{1}{a_2} \int \frac{-y_2(t)f(t)}{W[y_1, y_2](t)} dt = -4 \int e^{t/2} \sqrt{t} \sin(t/2)\, dt$$

and

$$v_2(t) = \frac{1}{a_2} \int \frac{y_1(t)f(t)}{W[y_1, y_2](t)} dt = 4 \int e^{t/2} \sqrt{t} \cos(t/2)\, dt.$$

The integrals are not easily figured out, so we leave them be. A particular solution $y_p(t)$ to the ODE is given by $v_1(t)y_1(t) + v_2(t)y_2(t)$, where $v_1(t)$ and $v_2(t)$ are unique up to arbitrary constant terms. One way to write out $y_p(t)$ is

$$y_p(t) = -4e^{-t/2} \cos(t/2) \int_{t_1}^{t} e^{\tau/2} \sqrt{\tau} \sin(\tau/2)\, d\tau + 4e^{-t/2} \sin(t/2) \int_{t_2}^{t} e^{\tau/2} \sqrt{\tau} \cos(\tau/2)\, d\tau.$$

Here t_1 and t_2 are arbitrary real numbers. The general solution is

$$y(t) = \big[c_1 \cos(t/2) + c_2 \sin(t/2)\big] e^{-t/2} + y_p(t),$$

with $y_p(t)$ being as given above.