Math 250 Exam #2 Key (Spring 2016)

1 Newton's Law of Cooling states that $T'(t) = k[T(t) - M]$, where M is the temperature of the oven. Here we have $T(0) = 70$, $T(0.5) = 120$, and $T(1) = 160$. Now,

$$T' = k(T - M) \;\Rightarrow\; \int \frac{dT}{T - M} = \int k\,dt \;\Rightarrow\; \ln|T - M| = kt + c \;\Rightarrow\; M - T = e^{kt+c},$$

and so

$$T(t) = M - Ce^{kt}.$$

From $T(0) = 70$ we obtain $70 = M - C$, so $C = M - 70$ and then

$$T(t) = M - (M - 70)e^{kt}.$$

From $T(0.5) = 120$ we obtain

$$120 = M - (M - 70)e^{0.5k} \;\Rightarrow\; e^{0.5k} = \frac{120 - M}{70 - M} \;\Rightarrow\; k = \ln\left(\frac{120 - M}{70 - M}\right)^2.$$

Thus

$$T(t) = M - (M - 70)\left(\frac{120 - M}{70 - M}\right)^{2t}$$

Now we use $T(1) = 160$ to get

$$160 = M - (70 - M)\left(\frac{120 - M}{70 - M}\right)^2,$$

which solves nicely to give $M = 320$°F.

2 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 5$. The volume of solution in the tank is $V(t) = 400 + 5t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned} x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\ &= \left(\frac{0.05\text{ kg}}{1\text{ L}}\right)\left(\frac{20\text{ L}}{1\text{ min}}\right) - \left(\frac{x(t)\text{ kg}}{V(t)\text{ L}}\right)\left(\frac{15\text{ L}}{1\text{ min}}\right) \\ &= 1 - \frac{15x(t)}{400 + 5t} = 1 - \frac{3x(t)}{80 + t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{3x}{t + 80} = 1.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{3}{t + 80}\,dt\right) = e^{3\ln(t+80)} = (t + 80)^3$$

to obtain

$$(t + 80)^3x' + 3(t + 80)^2x = (t + 80)^3,$$

which becomes

$$\left[(t + 80)^3x\right]' = (t + 80)^3$$

and thus

$$(t+80)^3 x = \int (t+80)^3\,dt = \frac{1}{4}(t+80)^4 + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{t}{4} + \frac{c}{(t+80)^3} + 20.$$

To determine c we use the initial condition $x(0) = 5$, giving $c = -15(80^3)$, and so

$$x(t) = \frac{t}{4} - 15\left(\frac{80}{t+80}\right)^3 + 20.$$

The amount of sugar in the tank after 1 hour (60 minutes) is

$$x(60) = \frac{60}{4} - 15\left(\frac{80}{140}\right)^3 + 20 \approx 32.2 \text{ kg}.$$

3 Suppose c_1, c_2, c_3 are constants such that $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$. That is,

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

for all $x \in \mathbb{R}$, and hence

$$c_1 x + c_2(6x-1) + c_3(2x+3) = 0$$

for all $x \in \mathbb{R}$. If $c_2 = 3$ and $c_3 = 1$ then we get

$$c_1 x + 3(6x-1) + (2x+3) = 0,$$

and hence $c_1 x + 20x = 0$. This last equation is satisfied on $(-\infty, \infty)$ if we let $c_1 = -20$. That is, $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$ is possible if we choose $c_1 = -20$, $c_2 = 3$, and $c_3 = 1$. Since $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$ admits a solution other than $c_1 = c_2 = c_3 = 0$, we conclude that f, g, and h are linearly dependent on $(-\infty, \infty)$.

4 The condition $y(0) = 3$ is satisfied by any member of the family. From $y(1) = 0$ we have $0 = c_1 + c_2 + 3$, so and $c_2 = -c_1 - 3$. Thus any member of the family of the form

$$y = cx^2 - (c+3)x^4 + 3,$$

where $c \in \mathbb{R}$ is arbitrary, will satisfy both boundary conditions.

5 The auxiliary equation $r^2 - 10r + 25 = 0$ has double root 5, and so the general solution is

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}.$$

6 The auxiliary equation is $r^4 + r^3 + r^2 = 0$, or $r^2(r^2 + r + 1) = 0$, which has double root 0 and complex roots $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. The general solution is thus

$$y(x) = c_1 + c_2 x + e^{-x/2}\left(c_3 \cos \tfrac{\sqrt{3}}{2}x + c_4 \sin \tfrac{\sqrt{3}}{2}x\right).$$

7 Auxiliary equation is $4r^2 - 4r - 3 = 0$, or $(2r+1)(2r-3) = 0$. Roots are $-\frac{1}{2}$ and $\frac{3}{2}$, so general solution is

$$y(x) = c_1 e^{-x/2} + c_2 e^{3x/2},$$

and so

$$y'(x) = -\tfrac{1}{2} c_1 e^{-x/2} + \tfrac{3}{2} c_2 e^{3x/2}.$$

From $y(0) = 1$ comes $1 = c_1 + c_2$, or $c_2 = 1 - c_1$. From $y'(0) = 5$ comes

$$5 = -\tfrac{1}{2} c_1 + \tfrac{3}{2} c_2 = -\tfrac{1}{2} c_1 + \tfrac{3}{2}(1 - c_1),$$

so $c_1 = -\frac{7}{4}$. Hence $c_2 = \frac{11}{4}$. Solution to initial value problem is

$$y(x) = -\frac{7}{4} e^{-x/2} + \frac{11}{4} e^{3x/2}.$$