1a Apply partial fraction decomposition:

$$\mathcal{L}^{-1}\left[\frac{5s}{(s-2)^2}\right] = 5\mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{2}{(s-2)^2}\right] = 5\left(e^{2t} + 2te^{2t}\right) = (10t+5)e^{2t}.$$

1b Apply $\mathcal{L}[g(t)u(t-a)] = e^{-as}\mathcal{L}[g(t+a)]$ with $g(t) = 3t+1$ and $a = 1$:

$$\mathcal{L}[(3t+1)u(t-1)] = e^{-s}\mathcal{L}[3(t+1)+1] = e^{-s}\mathcal{L}[3t+4] = e^{-s}\left(\frac{3}{s^2} + \frac{4}{s}\right).$$

1c Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right],$$

so that

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s}}{s^2+4} = e^{-\pi s}\mathcal{L}\left[\tfrac{1}{2}\sin 2t\right].$$

Now apply the formula

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\mathcal{L}[f(t)] \tag{1}$$

with $f(t) = \tfrac{1}{2}\sin 2t$ and $a = \pi$ to obtain

$$\mathcal{L}\left[\tfrac{1}{2}\sin 2(t-\pi)u(t-\pi)\right] = e^{-\pi s}\mathcal{L}\left[\tfrac{1}{2}\sin 2t\right] = \frac{e^{-\pi s}}{s^2+4},$$

and therefore

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right] = \tfrac{1}{2}\sin 2(t-\pi)u(t-\pi) = \tfrac{1}{2}\sin(2t)u(t-\pi).$$

2 We have $y' + 2y = t - tu(t-1)$. Let $Y = \mathcal{L}[y]$, which is a function of s. Taking the Laplace transform of the equation yields

$$[sY - y(0)] + 2Y = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right),$$

so that

$$Y = \frac{1}{s^2(s+2)} - e^{-s}\left(\frac{s+1}{s^2(s+2)}\right),$$

and thus

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2(s+2)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}(s+1)}{s^2(s+2)}\right] \tag{2}$$

By partial fraction decomposition we have

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2},$$

so that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+2)}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = -\tfrac{1}{4} + \tfrac{1}{2}t + \tfrac{1}{4}e^{-2t}.$$

The other term in (2) is handled similarly, giving
$$y(t) = -\tfrac{1}{4} + \tfrac{1}{2}t + \tfrac{1}{4}e^{-2t} - \tfrac{1}{4}u(t-1) - \tfrac{1}{2}(t-1)u(t-1) + \tfrac{1}{4}e^{-2(t-1)}u(t-1).$$

3 Taking the Laplace transform of the equation gives
$$\mathcal{L}[f(t)] = \mathcal{L}\big[te^t\big] + \mathcal{L}\bigg[\int_0^t \tau f(t-\tau)d\tau\bigg]$$
$$= \mathcal{L}\big[te^t\big] + \mathcal{L}[t]\mathcal{L}[f(t)] = \frac{1}{(s-1)^2} + \frac{1}{s^2}\mathcal{L}[f(t)],$$

and hence
$$\mathcal{L}[f(t)] = \frac{s^2}{(s+1)(s-1)^3} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}.$$

Applying the partial fraction decomposition method, we obtain
$$\mathcal{L}[f(t)] = \frac{-1/8}{s+1} + \frac{1/8}{s-1} + \frac{3/4}{(s-1)^2} + \frac{1/2}{(s-1)^3},$$

so that
$$f(t) = \mathcal{L}^{-1}\bigg[\frac{-1/8}{s+1}\bigg] + \mathcal{L}^{-1}\bigg[\frac{1/8}{s-1}\bigg] + \mathcal{L}^{-1}\bigg[\frac{3/4}{(s-1)^2}\bigg] + \mathcal{L}^{-1}\bigg[\frac{1/2}{(s-1)^3}\bigg],$$

and finally
$$f(t) = -\tfrac{1}{8}e^{-t} + \tfrac{1}{8}e^t + \tfrac{3}{4}te^t + \tfrac{1}{4}t^2 e^t.$$

4 We take the Laplace transform of the ODE:
$$\mathcal{L}[y''] + 2\mathcal{L}[y'] = \mathcal{L}[\delta(t-1)].$$

Now, letting $Y = \mathcal{L}[y]$ we obtain
$$[s^2 Y - sy(0) - y'(0)] + [sY - y(0)] = e^{-s} \quad \Rightarrow \quad (s^2 Y - 1) + sY = e^{-s}$$

whence
$$Y = \frac{1 + e^{-s}}{s^2 + s}$$

and so
$$y(t) = \mathcal{L}^{-1}\bigg[\frac{1}{s^2 + s}\bigg] + \mathcal{L}^{-1}\bigg[\frac{e^{-s}}{s^2 + s}\bigg]. \tag{3}$$

Partial fractions decomposition gives
$$\mathcal{L}^{-1}\bigg[\frac{1}{s^2 + s}\bigg] = \mathcal{L}^{-1}\bigg[\frac{1}{s} - \frac{1}{s+1}\bigg] = 1 - e^{-t}. \tag{4}$$

Using (4), and also the formula (1) with $f(t) = 1 - e^{-t}$ and $a = 1$, we get
$$\mathcal{L}\big[(1 - e^{-(t-1)})u(t-1)\big] = e^{-s}\mathcal{L}\big[1 - e^{-t}\big] = \frac{e^{-s}}{s^2 + s},$$

and therefore (3) becomes
$$y(t) = 1 - e^{-t} + (1 - e^{1-t})u(t-1).$$