Math 250 Exam #4 Key (Spring 2015)

1 We have

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 5\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - 5\sum_{n=2}^{\infty} c_{n-2}x^n$$

$$= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=2}^{\infty} 5c_{n-2}x^n$$

$$= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}\big[(n+1)(n+2)c_{n+2} - 5c_{n-2}\big]x^n.$$

2 Substituting

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

into the ODE gives

$$(2x-4)\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0,$$

which with reindexing becomes

$$\sum_{n=0}^{\infty}\big[(2n+1)c_n - 4(n+1)c_{n+1}\big]x^n = 0.$$

This implies that

$$(2n+1)c_n - 4(n+1)c_{n+1} = 0$$

for all $n \geq 0$, so in particular

$$c_1 = \frac{1}{4}c_0, \quad c_2 = \frac{3}{4^2 \cdot 2!}c_0, \quad c_3 = \frac{3\cdot 5}{4^3 \cdot 3!}c_0, \quad c_4 = \frac{3\cdot 5\cdot 7}{4^4\cdot 4!}c_0,$$

and in general

$$c_n = \frac{3\cdot 5\cdot 7\cdots(2n-1)}{4^n \cdot n!}c_0,$$

so that

$$y = c_0\sum_{n=0}^{\infty}\frac{3\cdot 5\cdot 7\cdots(2n-1)}{4^n\cdot n!}x^n = c_0\sum_{n=0}^{\infty}\frac{2(2n-1)!}{(n-1)!n!}\left(\frac{x}{8}\right)^n.$$

3 Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + 8\sum_{n=0}^{\infty} c_n x^n = 0,$$

whence comes

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=0}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} 8c_n x^n = 0,$$

and then

$$\sum_{n=0}^{\infty}\big[(n+1)(n+2)c_{n+2}-2nc_n+8c_n\big]x^n=0.$$

This implies that

$$(n+1)(n+2)c_{n+2}-2nc_n+8c_n=0,$$

for all $n\geq 0$, and hence

$$c_{n+2}=\frac{2n-8}{(n+1)(n+2)}c_n.$$

We now calculate

$$c_2=-4c_0,\quad c_3=\frac{-6}{3!}c_1,\quad c_4=\frac{4}{3}c_0,\quad c_5=\frac{(-6)(-2)}{5!}c_1,\quad c_6=0,\quad c_7=\frac{(-6)(-2)(2)}{7!}c_1,$$

$$c_8=0,\quad c_9=\frac{(-6)(-2)(2)(6)}{9!}c_1,\quad c_{10}=0,$$

and in general

$$c_{2n+1}=\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n+1)!}c_1$$

for $n\geq 0$, and $c_{2n}=0$ for $n\geq 3$. Now, since

$$y=\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}c_{2n}x^{2n}+\sum_{n=0}^{\infty}c_{2n+1}x^{2n+1},$$

we conclude that

$$y=c_0\left(1-4x^2+\frac{4}{3}x^4\right)+c_1\left(x+\sum_{n=1}^{\infty}\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n+1)!}x^{2n+1}\right).$$

This along with the initial condition $y(0)=3$ yields $c_0=3$. From

$$y'=-8c_0x+\frac{16}{3}c_0x^3+c_1\left(1+\sum_{n=1}^{\infty}\frac{(-6)(-2)(2)\cdots(4n-10)}{(2n)!}x^{2n}\right)$$

and the initial condition $y'(0)=0$ we get $c_1=0$. Therefore

$$y=3\left(1-4x^2+\frac{4}{3}x^4\right)=3-12x^2+4x^4$$

is the solution to the IVP.

4 We have

$$\mathcal{L}[f](s)=\int_0^2 e^{-st}dt+\int_2^{\infty}te^{-st}dt=-\frac{1}{s}(e^{-2s}-1)+\left(\frac{2}{s}+\frac{1}{s^2}\right)e^{-2s}=\frac{1}{s}+\left(\frac{1}{s}+\frac{1}{s^2}\right)e^{-2s}.$$

5 Applying linearity properties,

$$\begin{aligned}\mathcal{L}[f](s)&=\mathcal{L}\big[8t^3-12t^2+6t-1\big](s)=8\mathcal{L}\big[t^3\big](s)-12\mathcal{L}\big[t^2\big](s)+6\mathcal{L}[t](s)-\mathcal{L}[1](s)\\&=8\cdot\frac{3!}{s^4}-12\cdot\frac{2!}{s^3}+6\cdot\frac{1!}{s^2}-\frac{0!}{s}=\frac{48}{s^4}-\frac{24}{s^3}+\frac{6}{s^2}-\frac{1}{s}.\end{aligned}$$

6 Applying partial fraction decomposition and linearity properties,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s+1}{s^2-4s}\right](t) &= \mathcal{L}^{-1}\left[\frac{s+1}{s(s-4)}\right](t) = \mathcal{L}^{-1}\left[\frac{-1/4}{s}+\frac{5/4}{s-4}\right](t)\\ &= -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right](t)+\frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) = -\frac{1}{4}+\frac{5}{4}e^{4t}.\end{aligned}$$

7 Let $Y=\mathcal{L}[y](s)$, so the Laplace transform of the ODE is

$$\mathcal{L}[y'](s)-\mathcal{L}[y](s)=2\mathcal{L}[\cos 5t](s) \quad\Rightarrow\quad sY-y(0)-Y=\frac{2s}{s^2+25} \quad\Rightarrow\quad Y=\frac{2s}{(s-1)(s^2+25)},$$

and hence

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left[\frac{2s}{(s-1)(s^2+25)}\right](t) = \mathcal{L}^{-1}\left[\frac{\frac{1}{13}}{s-1}+\frac{\frac{25}{13}-\frac{1}{13}s}{s^2+25}\right](t)\\ &= \frac{1}{13}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t)-\frac{1}{13}\left(\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right](t)-5\mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right](t)\right)\\ &= \frac{1}{13}e^t-\frac{1}{13}\big(\cos 5t-5\sin 5t\big)=\frac{e^t-\cos 5t+5\sin 5t}{13}.\end{aligned}$$