Math 250 Exam #3 Key (Spring 2015)

1a We have $y'' + y' + 4y = e^t + e^{-t}$.

$$\text{Auxiliary equation: } r^2 + r + 4 = 0; \quad \text{roots: } -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i.$$

Start with equation $y'' + y' + 4y = e^t$, with nonhomogeneity $f(t) = e^t$. Since $\alpha = 1$ is not a root of the auxiliary equation, a particular solution has form $y_1(t) = Ae^t$. Substituting into ODE:

$$Ae^t + (Ae^t)' + 4(Ae^t)'' = e^t \quad \Rightarrow \quad 6Ae^t = e^t \quad \Rightarrow \quad A = \frac{1}{6},$$

so $y_1(t) = \frac{1}{6}e^t$.

Next we have $y'' + y' + 4y = e^{-t}$, with nonhomogeneity $f(t) = e^{-t}$. Since $\alpha = -1$ is not a root of the auxiliary equation, a particular solution has form $y_2(t) = Ae^{-t}$. Substituting into ODE:

$$Ae^{-t} + (Ae^{-t})' + 4(Ae^{-t})'' = e^{-t} \quad \Rightarrow \quad 4Ae^{-t} = e^{-t} \quad \Rightarrow \quad A = \frac{1}{4},$$

so $y_2(t) = \frac{1}{4}e^{-t}$.

By Superposition Principle a particular solution to the original equation is

$$y_p(t) = \frac{1}{6}e^t + \frac{1}{4}e^{-t}.$$

The general solution is thus

$$y(t) = \frac{1}{6}e^t + \frac{1}{4}e^{-t} + e^{-t/2}\left(c_1 \cos\frac{\sqrt{15}}{2}t + c_2 \sin\frac{\sqrt{15}}{2}t\right).$$

1b The auxiliary equation $r^2 + 2r + 5 = 0$ has roots $-1 \pm 2i$. The nonhomogeneity has the factor $e^{\alpha t}\cos\beta t$ with $\alpha - 1$ and $\beta = 2$, and since $\alpha + \beta i = -1 + 2i$ is a root of the auxiliary equation, we take $s = 1$ in the Method of Undetermined Coefficients. Thus a particular solution has form

$$y_p(t) = Ate^{-t}\cos 2t + Bte^{-t}\sin 2t.$$

Now $y_p''(t) + 2y_p'(t) + 5y_p(t) = 4e^{-t}\cos 2t$ becomes

$$(-4A\sin 2t + 4B\cos 2t)e^{-t} = (4\cos 2t)e^{-t},$$

which shows that $A = 0$ and $B = 1$, and therefore

$$y_p(t) = te^{-t}\sin 2t.$$

The general solution is thus

$$y(t) = te^{-t}\sin 2t + e^{-t}(c_1\cos 2t + c_2 \sin 2t). \tag{1}$$

Next, substituting the initial condition $y(0) = 1$ into (1) yields $c_1 = 1$. Substituting the initial condition $y'(0) = 0$ into the derivative of (1) yields $c_2 = \frac{1}{2}$. We now obtain the solution to the IVP:

$$y(t) = te^{-t}\sin 2t + e^{-t}\left(\cos 2t + \tfrac{1}{2}\sin 2t\right)$$

2 The auxiliary equation $r^2 - 2r + 1 = 0$ has double root 1, so the corresponding homogeneous equation $y'' - 2y' + 1 = 0$ has linearly independent solutions $y_1(t) = e^t$ and $y_2(t) = te^t$, and thus

$$y_1'(t) = e^t \quad \text{and} \quad y_2'(t) = (1+t)e^t.$$

Now, since $a_2 = 1$ and $f(t) = (1+t^2)^{-1}e^t$, we have

$$v_1(t) = \int \frac{-te^t \cdot (1+t^2)^{-1}e^t}{e^{2t}}dt = -\int \frac{t}{1+t^2}dt = -\frac{1}{2}\ln(1+t^2),$$

and

$$v_2(t) = \int \frac{e^t \cdot (1+t^2)^{-1}e^t}{e^{2t}}dt = \int \frac{1}{1+t^2}dt = \tan^{-1}(t).$$

Hence a particular solution to the ODE is

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = -\tfrac{1}{2}e^t\ln(1+t^2) + te^t\tan^{-1}(t).$$

General solution is therefore

$$y(t) = \left[-\tfrac{1}{2}\ln(1+t^2) + t\tan^{-1}(t) + c_1 t + c_2\right]e^t.$$

3a By Hooke's Law $F = ky$ we calculate the spring constant k as $k = F/y = 64/0.32 = 200$ lb/ft. Mass m and weight W are related by $W = mg$, and so mass is $m = W/g = 64/32 = 2$ slugs. There is no frictional force mentioned, so the system is assumed to be undamped. The IVP that models the mass-spring system is

$$2y'' + 200y = 0, \quad y(0) = -\tfrac{2}{3}\text{ ft}, \quad y'(0) = 5\text{ ft/s}.$$

Note it is necessary to convert inches to feet! The auxiliary equation $2r^2 + 200 = 0$ has solutions $r = \pm 10i$, and so the solution to the ODE is

$$y(t) = c_1\cos 10t + c_2\sin 10t.$$

With the initial conditions we determine the solution to the IVP to be

$$y(t) = -\tfrac{2}{3}\cos 10t + \tfrac{1}{2}\sin 10t.$$

3b From

$$y'(t) = \tfrac{20}{3}\sin 10t + 5\cos 10t \quad \text{and} \quad y''(t) = \tfrac{200}{3}\cos 10t - 50\sin 10t$$

we reckon

$$y(3) = -0.60\text{ ft}, \quad y'(3) = -5.82\text{ ft/s}, \quad y''(3) = 59.69\text{ ft/s}^2.$$

3c The period is $P = \pi/5$ seconds and the amplitude is

$$A = \sqrt{\left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \tfrac{5}{6}\text{ ft}.$$

3d We must first find the times t for which $y(t) = 0$. We have

$$y(t) = 0 \;\Rightarrow\; \tfrac{1}{2}\sin 10t = \tfrac{2}{3}\cos 10t \;\Rightarrow\; \tan 10t = \tfrac{4}{3} \;\Rightarrow\; t = \arctan\left(\tfrac{4}{3}\right).$$

Thus the object is at the equilibrium position at time $t = \arctan\left(\tfrac{4}{3}\right) + \tfrac{\pi}{10}n$, where $n \geq 0$ is any integer. When $n = 0$ we obtain $\arctan\left(\tfrac{4}{3}\right) \approx 0.0927$ second as the first time. The velocity at this time is

$$y'(0.0927) = \tfrac{20}{3}\sin(0.927) + 5\cos(0.927) \approx 8.33\text{ ft/s}.$$

The velocity at successive times will alternate between -8.33 ft/s and 8.33 ft/s.