

MATH 250 EXAM #2 KEY (SPRING 2015)

1 Newton's Law of Cooling states that $T'(t) = k[T(t) - T_a]$. Here we have $T_a = 5$, $T(1) = 55$, and $T(5) = 30$. Now, noting that $T(t) \geq 5$ for all $t \geq 0$,

$$T' = k(T - 5) \Rightarrow \int \frac{1}{T - 5} dT = \int k dt \Rightarrow \ln|T - 5| = kt + c \Rightarrow T - 5 = e^{kt+c},$$

and so

$$T(t) = 5 + Ce^{kt}.$$

From $T(1) = 55$ we obtain

$$55 = 5 + Ce^k \Rightarrow Ce^k = 50 \Rightarrow C = 50e^{-k},$$

and so

$$T(t) = 5 + 50e^{-k}e^{kt} = 5 + 50e^{k(t-1)}.$$

From $T(5) = 30$ we obtain

$$30 = 5 + 50e^{4k} \Rightarrow e^{4k} = \frac{1}{2} \Rightarrow 4k = \ln\left(\frac{1}{2}\right) \Rightarrow k = \frac{1}{4} \ln\left(\frac{1}{2}\right) \approx -0.173.$$

Thus

$$T(t) = 5 + 50e^{(t-1)\ln(0.5^{1/4})} = 5 + 50\left(\frac{1}{2}\right)^{(t-1)/4} = 5 + 50\sqrt[4]{\frac{1}{2^{t-1}}}.$$

The temperature in the house is given by

$$T(0) = 5 + 50\sqrt[4]{\frac{1}{2^{0-1}}} \approx 64.5^\circ\text{F}.$$

2 Let $x(t)$ be the mass of sugar (in kilograms) in the tank at time t (in minutes), so that $x(0) = 5$. The volume of solution in the tank is $V(t) = 400 + 5t$. The rate of change of the amount of sugar in the tank at time t is:

$$\begin{aligned} x'(t) &= (\text{rate sugar enters Tank 1}) - (\text{rate sugar leaves Tank 1}) \\ &= \left(\frac{0.05 \text{ kg}}{1 \text{ L}}\right)\left(\frac{20 \text{ L}}{1 \text{ min}}\right) - \left(\frac{x(t) \text{ kg}}{V(t) \text{ L}}\right)\left(\frac{15 \text{ L}}{1 \text{ min}}\right) \\ &= 1 - \frac{15x(t)}{400 + 5t} = 1 - \frac{3x(t)}{80 + t}. \end{aligned}$$

Thus we have a linear first-order ODE:

$$x' + \frac{3x}{t + 80} = 1.$$

To solve this equation, we multiply by the integrating factor

$$\mu(t) = \exp\left(\int \frac{3}{t + 80} dt\right) = e^{3\ln(t+80)} = (t + 80)^3$$

to obtain

$$(t + 80)^3 x' + 3(t + 80)^2 x = (t + 80)^3,$$

which becomes

$$[(t + 80)^3 x]' = (t + 80)^3$$

and thus

$$(t + 80)^3 x = \int (t + 80)^3 dt = \frac{1}{4}(t + 80)^4 + c.$$

From this we get a general explicit solution to the ODE,

$$x(t) = \frac{t}{4} + \frac{c}{(t + 80)^3} + 20.$$

To determine c we use the initial condition $x(0) = 5$, giving $c = -15(80^3)$, and so

$$x(t) = \frac{t}{4} - 15 \left(\frac{80}{t + 80} \right)^3 + 20.$$

The amount of sugar in the tank after 30 minutes is

$$x(30) = \frac{30}{4} - 15 \left(\frac{80}{110} \right)^3 + 20 \approx 21.7 \text{ kg.}$$

3 Suppose c_1, c_2, c_3 are constants such that $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$. That is,

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

for all $x \in \mathbb{R}$, and hence

$$c_1 x + c_2(x - 1) + c_3(x + 3) = 0$$

for all $x \in \mathbb{R}$. If $c_2 = 3$ and $c_3 = 1$ then we get

$$c_1 x + 3(x - 1) + (x + 3) = 0,$$

and hence $c_1 x + 4x = 0$. This last equation is satisfied on $(-\infty, \infty)$ if we let $c_1 = -4$. That is, $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$ is possible if we choose $c_1 = -4$, $c_2 = 3$, and $c_3 = 1$. Since $c_1 f + c_2 g + c_3 h \equiv 0$ on $(-\infty, \infty)$ admits a solution other than $c_1 = c_2 = c_3 = 0$, we conclude that f , g , and h are linearly dependent on $(-\infty, \infty)$.

4 We have $y(x) = c_1 + c_2 x^2$ and $y'(x) = 2c_2 x$. From $y(0) = 1$ we get $1 = c_1 + c_2(0^2) = c_1$, or $c_1 = 1$; and from $y'(1) = 6$ we get $6 = 2c_2(1)$, or $c_2 = 3$. Therefore a solution to the boundary value problem is $y = 1 + 3x^2$.

5 A second solution y_2 will be of the form

$$y_2(x) = u(x)y_1(x) = xu(x) \sin(\ln x),$$

which we'll write simply as $y = xu \sin(\ln x)$. Now,

$$y' = xu \cdot \frac{\cos(\ln x)}{x} + (u + xu') \cdot \sin(\ln x) = u \cos(\ln x) + u \sin(\ln x) + xu' \sin(\ln x),$$

and

$$y'' = \frac{u \cos(\ln x) - u \sin(\ln x)}{x} + 2u' \cos(\ln x) + 2u' \sin(\ln x) + xu'' \sin(\ln x).$$

Oh dear. Oh deary me. Substituting all this into $x^2 y'' - xy' + 2y = 0$ gives

$$x[u \cos(\ln x) - u \sin(\ln x)] + x^2[2u' \cos(\ln x) + 2u' \sin(\ln x) + xu'' \sin(\ln x)]$$

$$-x[u \cos(\ln x) + u \sin(\ln x) + xu' \sin(\ln x)] + 2xu \sin(\ln x) = 0,$$

whence

$$x^3 \sin(\ln x)u'' + x^2[\sin(\ln x) + 2 \cos(\ln x)]u' = 0.$$

Letting $w = u'$ and dividing out $x^3 \sin(\ln x)$ gives

$$w' + \frac{\sin(\ln x) + 2 \cos(\ln x)}{x \sin(\ln x)}w = 0.$$

Another way to write this is

$$w' + \left(\frac{1}{x} + \frac{2 \cos(\ln x)}{x \sin(\ln x)}\right)w = 0.$$

This is separable:

$$-\int \frac{1}{w} dw = \int \left(\frac{1}{x} + \frac{2 \cos(\ln x)}{x \sin(\ln x)}\right) dx = \ln|x| + 2 \int \frac{\cos(\ln x)}{x \sin(\ln x)} dx. \quad (1)$$

Making the substitution $\alpha = \ln x$, and then the substitution $\beta = \sin \alpha$, we have

$$\int \frac{\cos(\ln x)}{x \sin(\ln x)} dx = \int \frac{\cos \alpha}{\sin \alpha} d\alpha = \int \frac{1}{\beta} d\beta = \ln|\beta| = \ln|\sin \alpha| = \ln|\sin(\ln x)|.$$

Now (1) becomes

$$-\ln|u'| = -\ln|w| = \ln(x) + 2 \ln|\sin(\ln x)| + c = \ln(x \sin^2(\ln x)) + c,$$

where $|x| = x$ since it's clear early on that we must have $x > 0$. Choosing $c = 0$, we then get

$$\frac{1}{|u'|} = x \sin^2(\ln x) \Rightarrow |u'| = \frac{1}{x \sin^2(\ln x)} \Rightarrow u'(x) = \pm \frac{\csc^2(\ln x)}{x}.$$

We can choose $u'(x) = -x^{-1} \csc^2(\ln x)$, so, making the substitution $\alpha = \ln x$ again, we have

$$u(x) = -\int \frac{\csc^2(\ln x)}{x} dx = -\int \csc^2 \alpha d\alpha = \cot(\alpha) + c = \cot(\ln x) + c.$$

Again choosing $c = 0$, we obtain $u(x) = \cot(\ln x)$, and so

$$y_2(x) = x \cot(\ln x) \sin(\ln x) = x \cos(\ln x)$$

is a second solution to the ODE. This solution is long, but can be made shorter by using the formula in the book.

6 The auxiliary equation $2r^2 + 2r + 1 = 0$ has roots $-\frac{1}{2} \pm \frac{1}{2}i$, and so the general solution is

$$y(x) = e^{-x/2}(c_1 \cos(x/2) + c_2 \sin(x/2)).$$

This makes up for Problem (5) a little bit.

7 The auxiliary equation $r^3 - 6r^2 + 12r - 8 = 0$. It can be seen by inspection that $r = 2$ is a root. With a little synthetic division,

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

we obtain the factorization $(r - 2)(r^2 - 4r + 4) = 0$, and thus $(r - 2)(r - 2)(r - 2) = 0$. That is, 2 is a root of the auxiliary equation with multiplicity 3. The general solution is thus

$$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x} = (c_1 + c_2x + c_3x^2)e^{2x}.$$

8 The given general solution results from an auxiliary equation having root 0 with multiplicity 2, and root 8 with multiplicity 1. That is, $r^2(r - 8) = 0$, or $r^3 - 8r^2 = 0$. A suitable ODE is

$$y''' - 8y'' = 0.$$