Math 250 Exam #5 Key (Spring 2014)

1 Solution is

$$z(t) = e^{-t} + e^{-2t} + \frac{1}{2}\left[e^{-3t} - 2e^{-2(t+1)} + e^{-(t+4)}\right]u(t-2)$$

2 We take the Laplace transform of each side of the ODE, using linearity properties to obtain

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = 2\mathcal{L}[\delta(t-\pi)](s) - \mathcal{L}[\delta(t-2\pi)](s).$$

Now, letting $Y(s) = \mathcal{L}[y](s)$ we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = 2e^{-\pi s} - e^{-2\pi s},$$

whence

$$Y(s) = \frac{2e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}$$

and so

$$y(t) = \mathcal{L}^{-1}\left[\frac{2e^{-\pi s}}{s^2+4}\right](t) - \mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+4}\right](t). \tag{1}$$

If we define $h(t) = \sin(2t)$, then

$$\mathcal{L}[h(t)](s) = \frac{2}{s^2+4},$$

so

$$\mathcal{L}[h(t-\pi)u(t-\pi)](s) = e^{-\pi s}\mathcal{L}[h(t)](s) = \frac{2e^{-\pi s}}{s^2+4}$$

and hence

$$\mathcal{L}^{-1}\left[\frac{2e^{-\pi s}}{s^2+4}\right](t) = h(t-\pi)u(t-\pi) = \sin(2t-2\pi)u(t-\pi) = \sin(2t)u(t-\pi).$$

In similar fashion we obtain

$$\mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+4}\right](t) = \frac{1}{2}h(t-2\pi)u(t-2\pi) = \frac{1}{2}\sin(2t-4\pi)u(t-2\pi) = \frac{1}{2}\sin(2t)u(t-2\pi).$$

Putting these results into (1) yields

$$y(t) = \sin(2t)\left[u(t-\pi) - \tfrac{1}{2}u(t-2\pi)\right].$$

3 Letting

$$F(s) = \frac{1}{s^2+4},$$

we see that $F(s) = \mathcal{L}[f](s)$ for $f(t) = \frac{1}{2}\sin 2t$. Using the identity

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)],$$

we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right](t) = \mathcal{L}^{-1}[F(s)F(s)](t) = (f*f)(t) = \int_0^t \frac{1}{2}\sin\bigl(2(t-u)\bigr)\cdot\frac{1}{2}\sin(2u)\,du$$

$$= \frac{1}{8}\int_0^t \big[\cos(2t-4u) - \cos(2t)\big]\,du = \frac{1}{8}\left[-\frac{1}{4}\sin(2t-4u) - u\cos(2t)\right]_0^t$$
$$= \frac{1}{16}\sin(2t) - \frac{1}{8}t\cos(2t).$$

4 We have $y''(t) = y^2(t) - 2y'(t) + t^2$, and so using the initial conditions we obtain

$$y''(0) = y^2(0) - 2y'(0) + 0^2 = 1^2 - 2(1) = -1.$$

Next, $y''' = 2yy' - 2y'' + 2t$, so

$$y'''(0) = 2(1)(1) - 2(-1) + 2(0) = 4.$$

Finally, from $y^{(4)} = 2yy'' + 2(y')^2 - 2y''' + 2$ we have

$$y^{(4)}(0) = 2(1)(-1) + 2(1)^2 - 2(4) + 2 = -6.$$

The nth-order Taylor polynomial for y with center 0 is

$$P_n(t) = \sum_{k=0}^{n} \frac{y^{(k)}(0)}{k!}t^k = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + \cdots + \frac{y^{(n)}(0)}{n!}t^n.$$

We have

$$P_4(t) = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \frac{y^{(4)}(0)}{4!}t^4$$
$$= 1 + t - \frac{1}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{4}t^4.$$

That is,

$$y(t) \approx 1 + t - \frac{1}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{4}t^4$$

for all t near 0.

5 Solution will be of the form

$$y(t) = \sum_{k=0}^{\infty} c_k t^k.$$

Substituting this into the ODE gives

$$\sum_{k=1}^{\infty} k c_k t^{k-1} - \sum_{k=0}^{\infty} c_k t^k = 0.$$

Reindexing, we obtain

$$\sum_{k=0}^{\infty}(k+1)c_{k+1}t^k - \sum_{k=0}^{\infty} c_k t^k = 0,$$

or equivalently

$$\sum_{k=0}^{\infty}\left[(k+1)c_{k+1} - c_k\right]t^k = 0.$$

This implies that $(k+1)c_{k+1} - c_k = 0$ for all $k \geq 0$, or $c_{k+1} = c_k/(k+1)$. From this we find that

$$c_1 = c_0, \quad c_2 = \frac{c_1}{2} = c_0 2!, \quad c_3 = \frac{c_2}{3} = \frac{c_0}{3!},$$

and in general $c_k = c_0/k!$. Therefore

$$y(t) = \sum_{k=0}^{\infty} \frac{c_0}{k!} t^k = c_0 \left(1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \cdots \right),$$

where c_0 is an arbitrary constant.

6 We expect to find a general solution of the form

$$y(t) = \sum_{k=0}^{\infty} c_k t^k.$$

Substituting this into the ODE yields

$$\sum_{k=2}^{\infty} k(k-1)c_k t^{k-2} + t^2 \sum_{k=1}^{\infty} k c_k t^{k-1} + t \sum_{k=0}^{\infty} c_k t^k = 0,$$

and thus

$$\sum_{k=2}^{\infty} k(k-1)c_k t^{k-2} + \sum_{k=1}^{\infty} k c_k t^{k+1} + \sum_{k=0}^{\infty} c_k t^{k+1} = 0.$$

Reindex to get

$$\sum_{k=0}^{\infty} (k+1)(k+2)c_{k+2} t^k + \sum_{k=2}^{\infty} (k-1)c_{k-1} t^k + \sum_{k=1}^{\infty} c_{k-1} t^k = 0,$$

and then

$$\left[2c_2 + 6c_3 t + \sum_{k=2}^{\infty} (k+1)(k+2)c_{k+2} t^k \right] + \sum_{k=2}^{\infty} (k-1)c_{k-1} t^k + \left[c_0 t + \sum_{k=2}^{\infty} c_{k-1} t^k \right] = 0.$$

Hence

$$2c_2 + (6c_3 + c_0)t + \sum_{k=2}^{\infty} \left[(k+1)(k+2)c_{k+2} + (k-1)c_{k-1} + c_{k-1} \right] t^k = 0,$$

which simplifies to become

$$2c_2 + (6c_3 + c_0)t + \sum_{k=2}^{\infty} \left[(k+1)(k+2)c_{k+2} + k c_{k-1} \right] t^k = 0.$$

This implies that $2c_2 = 0$, $6c_3 + c_0 = 0$, and

$$(k+1)(k+2)c_{k+2} + k c_{k-1} = 0$$

for all $k \geq 2$. That is, $c_2 = 0$, $c_3 = -c_0/6 = -c_0/(2 \cdot 3)$, and

$$c_{k+2} = \frac{-k}{(k+1)(k+2)} c_{k-1}$$

for $k \geq 2$. The recursion relation enables us to express all c_k exclusively in terms of c_0 and c_1:

$$c_4 = \frac{-2}{3\cdot 4}c_1 \qquad c_5 = \frac{-3}{4\cdot 5}c_2 = 0$$

$$c_6 = \frac{-4}{5\cdot 6}c_3 = \frac{4}{2\cdot 3\cdot 5\cdot 6}c_0 \qquad c_7 = \frac{-5}{6\cdot 7}c_4 = \frac{2\cdot 5}{3\cdot 4\cdot 6\cdot 7}c_1$$

$$c_8 = \frac{-6}{7\cdot 8}c_5 = 0 \qquad c_9 = \frac{-7}{8\cdot 9}c_6 = \frac{-4\cdot 7}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}c_0$$

$$c_{10} = \frac{-8}{9\cdot 10}c_7 = \frac{-2\cdot 5\cdot 8}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}c_1 \qquad c_{11} = \frac{-9}{10\cdot 11}c_8 = 0$$

$$c_{12} = \frac{-10}{11\cdot 12}c_9 = \frac{4\cdot 7\cdot 10}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdot 11\cdot 12}c_0$$

So we have

$$y(t) = c_0 + c_1 t - \frac{c_0}{2\cdot 3}t^3 - \frac{2c_1}{3\cdot 4}t^4 + \frac{4c_0}{2\cdot 3\cdot 5\cdot 6}t^6 + \frac{2\cdot 5c_1}{3\cdot 4\cdot 6\cdot 7}t^7 - \frac{4\cdot 7c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}t^9 - \frac{2\cdot 5\cdot 8c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}t^{10} + \frac{4\cdot 7\cdot 10c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdot 11\cdot 12}t^{12} + \cdots$$

Setting $c_0 = 0$ and $c_1 = 1$ yields the particular solution

$$\begin{aligned} y_1(t) &= t - \frac{2}{3\cdot 4}t^4 + \frac{2\cdot 5}{3\cdot 4\cdot 6\cdot 7}t^7 - \frac{2\cdot 5\cdot 8}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}t^{10} + \cdots \\ &= t - \frac{2^2}{4!}t^4 + \frac{2^2\cdot 5^2}{7!}t^7 - \frac{2^2\cdot 5^2\cdot 8^2}{10!}t^{10} + \cdots \\ &= t + \sum_{k=1}^{\infty} \frac{2^2\cdot 5^2\cdots(3k-1)^2}{(-1)^k(3k+1)!}t^{3k+1}, \end{aligned}$$

and setting $c_0 = 1$ and $c_1 = 0$ yields the particular solution

$$\begin{aligned} y_2(t) &= 1 - \frac{1}{2\cdot 3}t^3 + \frac{4}{2\cdot 3\cdot 5\cdot 6}t^6 - \frac{4\cdot 7}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}t^9 + \frac{4\cdot 7\cdot 10}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdot 11\cdot 12}t^{12} + \cdots \\ &= 1 - \frac{1}{3!}t^3 + \frac{4^2}{6!}t^6 - \frac{4^2\cdot 7^2}{9!}t^9 + \frac{4^2\cdot 7^2\cdot 10^2}{12!}t^{12} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{4^2\cdot 7^2\cdots(3k-2)^2}{(-1)^k(3k)!}t^{3k}. \end{aligned}$$

Since $y_1(t)$ and $y_2(t)$ are linearly independent, the general solution to the ODE may be expressed as

$$y(t) = d_1\left[t + \sum_{k=1}^{\infty} \frac{2^2\cdot 5^2\cdots(3k-1)^2}{(-1)^k(3k+1)!}t^{3k+1}\right] + d_2\left[1 + \sum_{k=1}^{\infty} \frac{4^2\cdot 7^2\cdots(3k-2)^2}{(-1)^k(3k)!}t^{3k}\right],$$

where d_1 and d_2 are arbitrary constants.